

# QUANTUM MATRICES BY PATHS

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**ABSTRACT.** We study, from a combinatorial viewpoint, the *quantized coordinate ring of  $m \times n$  matrices* over an infinite field  $\mathbb{K}$  (also called *quantum matrices*) and its torus-invariant prime ideals. The first part of this paper shows that this algebra, traditionally defined by generators and relations, can be seen as subalgebra of a quantum torus by using paths in a certain directed graph. Roughly speaking, we view each generator of quantum matrices as a sum over paths in the graph, each path being assigned an element of the quantum torus. The quantum matrices relations then arise naturally by considering intersecting paths. This viewpoint is closely related to Cauchon’s deleting-derivations algorithm.

The second part of this paper is to apply the paths viewpoint to the theory of torus-invariant prime ideals of quantum matrices. We prove a conjecture of Goodearl and Lenagan that all such prime ideals, when the quantum parameter  $q$  is a non-root of unity, have generating sets consisting of quantum minors. Previously, this result was known to hold only for  $\text{char}(\mathbb{K}) = 0$  and  $q$  transcendental over  $\mathbb{Q}$ . Our strategy is to show that the quantum minors in a given torus-invariant ideal form a Gröbner basis.

## 1. INTRODUCTION

There has been much interest over the past decade in understanding the structure of the prime and primitive spectra of various “quantum” algebras. Arguably the most important progress in this direction is the  $\mathcal{H}$ -Stratification Theory of Goodearl and Letzter [12] (see also [1]). Briefly, many noncommutative rings support a rational action of a torus  $\mathcal{H}$  which allows one to partition the prime spectrum of the ring into finitely many  $\mathcal{H}$ -strata, each  $\mathcal{H}$ -stratum homeomorphic (with respect to the usual Zariski topology) to the prime spectrum of a Laurent polynomial ring in finitely many commuting indeterminates, and each containing a unique  $\mathcal{H}$ -invariant prime ideal. Moreover, the primitive ideals of the algebra are precisely those that are maximal within their  $\mathcal{H}$ -stratum. Thus, an important first step towards understanding the prime and primitive spectra is to first study the  $\mathcal{H}$ -invariant prime ideals called  $\mathcal{H}$ -primes.

The *quantized coordinate ring of  $m \times n$  matrices* over an infinite field  $\mathbb{K}$ , also simply called  *$(m \times n)$  quantum matrices* and denoted  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ , has received particular attention as this algebra underlies many of the “classical” quantum groups, e.g., the quantum special and general linear groups,

the quantum Grassmannian, etc. Moreover, while this algebra has a seemingly simple structure (for example, it is an iterated Ore extension over the field  $\mathbb{K}$ ), many problems have proven difficult to resolve. In particular, the machineries employed to analyze  $\text{spec}(\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K})))$  have tended to use fairly sophisticated viewpoints from noncommutative ring theory and representation theory. One of the purposes of this paper is to develop a “blue collar” approach to quantum matrices using combinatorics. We will describe this approach later in this section.

The  $\mathcal{H}$ -stratification theory applies to quantum matrices in the generic case, i.e., when the parameter  $q$  is a non-root of unity, and so a natural problem is to find generating sets for the  $\mathcal{H}$ -primes. For  $2 \times 2$  quantum matrices, this problem is fairly straightforward, yet even the  $3 \times 3$  case required a significant amount of work by Goodearl and Lenagan [9, 10]. However, in all cases the generating sets consisted of *quantum minors* and so it was conjectured that this held true in general.

The next significant step in the program to understand the prime spectrum of  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$  is the *deleting derivations algorithm* of Cauchon [4, 5]. Roughly speaking, this procedure shows that in many cases, including quantum matrices, one may embed the set of  $\mathcal{H}$ -primes of a given quantum algebra into the set of  $\mathcal{H}$ -primes of a *quantum affine space*. This is convenient since quantum affine spaces are typically easy to handle thanks to results of Goodearl and Letzter [11]. For quantum matrices, this allows one to obtain a combinatorial description of  $\mathcal{H}$ -primes using what are now known in the quantum algebra community as *Cauchon diagrams*. Perhaps surprisingly, these diagrams arose independently in work of Postnikov [17] through his investigations of the *totally nonnegative Grassmannian*. In this context, Cauchon diagrams are called  $\mathbf{J}$ -diagrams (also  $\mathbf{Le}$ -diagrams) and have been investigated by several authors (see Lam and Williams [13] and Talaska [18] in particular). The connections between these two areas and Poisson geometry have been explored by Goodearl, Launois and Lenagan [8].

Using Cauchon’s theory, Launois [14, 15] was the first to prove Goodearl and Lenagan’s conjecture under the constraints  $\mathbb{K} = \mathbb{C}$  and  $q$  transcendental over  $\mathbb{Q}$ . This was later extended to any  $\mathbb{K}$  of characteristic zero [7]. Launois also described an algorithm to find the generators, but the calculations involved very quickly become unwieldy. A graph theoretic interpretation of Launois’ algorithm was provided by the author of the current paper [3]. This latter work forms the starting point for some of the results presented below. In fact, much of Section 3 can be interpreted as a combinatorial interpretation of Cauchon and Launois’ work. Section 2 describes their work, as well as the  $\mathcal{H}$ -stratification theory.

Finally, let us also mention that Yakimov [20, 19] has developed representation theoretic methods to great success. In particular, he has independently verified (and generalized) Goodearl and Lenagan’s conjecture, but again, only under the constraint that  $\text{char}(\mathbb{K}) = 0$  and  $q$  transcendental over  $\mathbb{Q}$ . Furthermore, the generated sets Yakimov obtains are smaller than

Launois'. It is unclear how Yakimov's work relates to the viewpoint used in this paper, however, recent work of Geiger and Yakimov [6] explore the connections between Yakimov's work and Cauchon's, and so there is quite likely a close relationship. It should be added that Goodearl and Lenagan also conjectured that should a generating set of a given  $\mathcal{H}$ -prime consist of quantum minors, then these minors can be arranged to form a polynormal sequence (see [1]). Yakimov also proved this part of the conjecture under the above constraints on  $\mathbb{K}$  and  $q$ , and additionally showed that if  $q$  not a root of unity, then there exists a polynormal generating set, but one that may not consist of quantum minors. We do not address this part of the Goodearl-Lenagan conjecture in this paper.

As noted above, the approach we take towards quantum matrices is combinatorial. Recall that the usual description of  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$  is by generators and relations. Section 3 begins by giving a directed graph whose edges are "weighted" by elements of a quantum torus. Using these weights, we ascribe to a directed path in the graph an element of the quantum torus. We may then discuss various subalgebras of the quantum torus generated by sums over paths weights. The main result of Section 3 is that quantum matrices can be obtained in this manner. In fact, we do a bit more: given an  $\mathcal{H}$ -prime  $K$ , we show that  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))/K$  is also such a subalgebra. This provides a nice combinatorial framework to understand the  $\mathcal{H}$ -primes.

That the "paths" view of quantum matrices is more than a mere curiosity is demonstrated by the bulk of this paper, Section 4. The main result here is Corollary 4.5.1, a proof of Goodearl and Lenagan's conjecture for any infinite field  $\mathbb{K}$  and  $q \in \mathbb{K}^*$  not a root of unity. Moreover, the generating sets we obtain are, in general, smaller than required in the conjecture. In Section 4.1, we recall the notion of quantum minors and the results of [3] that allow us to visualize a quantum minor as a sum over sets of vertex-disjoint paths. Our main result will follow from Theorem 4.4.1 in which we show that the quantum minors in a given  $\mathcal{H}$ -prime form a Gröbner basis.

## 2. QUANTUM MATRICES

This section reviews the relevant background needed for this paper. Some of the basic theory of quantum matrices and closely related algebras is presented in Section 2.1. We then describe how Cauchon's deleting derivations algorithm (Section 2.2) and the  $\mathcal{H}$ -stratification theory (Section 2.3) apply. Also in Section 2.3, we introduce the notion of *Cauchon diagrams* which play a fundamental role throughout this paper.

We first set some data, notation and conventions that are used throughout this paper.

- Fix: an infinite field  $\mathbb{K}$ , integers  $m, n \geq 2$ , and a nonzero, non-root of unity  $q \in \mathbb{K}$ .
- For a positive integer  $k$ , we set  $[k] = \{1, 2, \dots, k\}$ .

- The set of  $m \times n$  matrices with integer entries is denoted by  $\mathcal{M}_{m,n}(\mathbb{Z})$ . The subset of  $\mathcal{M}_{m,n}(\mathbb{Z})$  consisting of matrices with non-negative entries is denoted by  $\mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ .
- The  $(i, j)$ -entry of a matrix  $N \in \mathcal{M}_{m,n}(\mathbb{Z})$  is denoted by  $(N)_{i,j}$ , and  $(i, j)$  is called the *coordinate* of this entry. In view of this, the elements of  $[m] \times [n]$  are called coordinates.
- We may describe relative positions of entries or coordinates using common terms such as north, northwest etc. For example,  $(i, j)$  is *northwest* of  $(r, s)$  if  $i < r$  and  $j < s$ , and *north* if  $j = s$ .
- We omit braces when taking the union of a set and a single element or removing a single element from a set. For example, we write  $I \cup \{i\} = I \cup i$  and  $I \setminus \{i\} = I \setminus i$ .

**Remark 2.0.1.** The restriction  $m, n \geq 2$  is made simply to avoid some inconveniences in various definitions that would occur if  $m = 1$  or  $n = 1$ . Fortunately, it is already known that all results presented in this paper hold when  $m = 1$  or  $n = 1$  since in these cases all algebras of this paper reduce to a certain quantum affine space, and such algebras can be dealt with using results of [11].

### 2.1. The Algebras $R^{(t)}$ .

**Definition 2.1.1.** The *lexicographic order* on  $[m] \times [n]$  is the total order  $<$  obtained by setting

$$(i, j) < (k, \ell) \Leftrightarrow i < k, \text{ or, } i = k \text{ and } j < \ell.$$

If  $(i, j) \in [m] \times [n]$ , then  $(i, j)^-$  denotes the largest element less than  $(i, j)$  with respect to the lexicographic order.

For example, the lexicographic order on  $[2] \times [3]$  satisfies

$$(1, 1) < (1, 2) < (1, 3) < (2, 1) < (2, 2) < (2, 3).$$

The symbol used for the lexicographic order on  $[m] \times [n]$  is, of course, an abuse of the usual meaning of  $<$  with respect to integers. We will use both meanings in this paper, the intended of which will be clear from the context.

**Note 2.1.2.** All references in this paper relating to an ordering of  $[m] \times [n]$  are with respect to the lexicographic order.

**Remark 2.1.3.** While the choice of lexicographic order suffices for our purposes, it is not quite necessary. To explain, suppose we index, in the usual manner, the squares of an  $m \times n$  grid using  $[m] \times [n]$ . Then all results of this paper hold, with appropriate modification, using any total order on  $[m] \times [n]$  that satisfies the property that for every positive integer  $t \in [mn]$ , the  $t$  smallest elements of  $[m] \times [n]$  with respect to this order form a Young diagram.

The lexicographic order is here fixed so as to simplify notation, to match the ordering used in the literature, and since there does not yet seem to be any advantage in considering a different order.

The algebras in the next definition each have a set of generators indexed by the coordinates of  $[m] \times [n]$ . It is convenient to place these generators as the entries of an  $m \times n$  matrix that we call the *matrix of (standard) generators*.

**Definition 2.1.4.** Let  $t \in [mn]$  and set  $(r, s)$  to be the  $t^{\text{th}}$  smallest coordinate. Define  $R^{(t)}$  to be the  $\mathbb{K}$ -algebra with the  $m \times n$  matrix of generators  $X = [x_{i,j}]$  subject to the following relations. If

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is any  $2 \times 2$  submatrix of  $X$ , then:

- (1)  $ab = qba$ ,  $cd = qdc$ ;
- (2)  $ac = qca$ ,  $bd = qdb$ ;
- (3)  $bc = cb$ ;
- (4)  $ad = \begin{cases} da, & \text{if } d = x_{k,\ell} \text{ and } (k, \ell) > (r, s); \\ da + (q - q^{-1})bc, & \text{if } d = x_{k,\ell} \text{ and } (k, \ell) \leq (r, s). \end{cases}$

**Example 2.1.5.** If  $m = 2$ ,  $n = 3$  and  $t = 5$ , then  $(r, s) = (2, 2)$  and  $R^{(5)}$  has matrix of generators

$$\begin{bmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \end{bmatrix}.$$

The relations corresponding to Part 4 of Definition 2.1.4 are

$$\begin{aligned} x_{1,1}x_{2,2} &= x_{2,2}x_{1,1} + (q - q^{-1})x_{1,2}x_{2,1} \\ x_{1,1}x_{2,3} &= x_{2,3}x_{1,1} \\ x_{1,2}x_{2,3} &= x_{2,3}x_{1,2}. \end{aligned}$$

The two extremities in the collection of  $R^{(t)}$  are of the most interest to us.

**Notation 2.1.6.** With respect to the notation in Definition 2.1.4:

- (1) If  $t = 1$ , then in Part 4 of Definition 2.1.4 we always have

$$ad = da.$$

We call this algebra  $m \times n$  *quantum affine space*, which we also denote by  $\mathcal{O}_q(\mathbb{K}^{m \times n})$ . The standard generators of  $\mathcal{O}_q(\mathbb{K}^{m \times n})$  will usually be labeled by  $t_{i,j}$  for  $(i, j) \in [m] \times [n]$ .

- (2) If  $t = mn$ , then in Part 4 of Definition 2.1.4 we always have

$$ad = da + (q - q^{-1})bc.$$

This algebra is the *quantized coordinate ring of  $m \times n$  matrices over  $\mathbb{K}$* , denoted by  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$  and simply referred to as  $(m \times n)$  *quantum matrices*.

- (3) The localization of  $R^{(1)} = \mathcal{O}_q(\mathbb{K}^{m \times n})$  with respect to the multiplicative set generated by the standard generators  $t_{i,j}$  is called the  $(m \times n)$  quantum torus  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ .

In later sections, we will need to intimately work with monomials in the standard generators of  $R^{(t)}$ , so we here set some notation in this respect. For the remainder of this section, fix  $t \in [mn]$  and let  $[x_{i,j}]$  be the matrix of standard generators for  $R^{(t)}$ .

**Notation 2.1.7.** If  $N \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ , then we write

$$\mathbf{x}^N = x_{1,1}^{(N)_{1,1}} x_{1,2}^{(N)_{1,2}} \cdots x_{m,n}^{(N)_{m,n}} \in R^{(t)}.$$

If  $(r, s)$  is the  $t^{\text{th}}$  smallest coordinate, then similar notation will be used when the  $t^{\text{th}}$  entry of  $N$  is negative, in which case  $\mathbf{x}^N \in R^{(t)}[x_{r,s}^{-1}]$ . It is essential to notice here that the generators in  $\mathbf{x}^N$  are always written from left to right so that the indices obey the lexicographic order from smallest to largest. Such an ordered monomial will be called a *lexicographic term*.

It is not difficult to write each  $R^{(t)}$  as an iterated Ore extension which immediately gives some nice properties.

**Theorem 2.1.8.** *The following properties hold for every  $t \in [mn]$ .*

- (1)  $R^{(t)}$  is a Noetherian domain.
- (2) As a  $\mathbb{K}$ -vector space,  $R^{(t)}$  has a basis consisting of the lexicographic terms  $\mathbf{x}^N$  with  $N \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ . The same properties also hold for the  $m \times n$  quantum torus (but with  $N \in \mathcal{M}_{m,n}(\mathbb{Z})$ ).  $\square$

**Definition 2.1.9.** For  $a \in R^{(t)}$ , the (unique) linear combination of distinct lexicographic terms with nonzero coefficients that equals  $a$  is called the *lexicographic expression* of  $a$ . We will say that a lexicographic term *appears* in  $a$  if it is a term in the lexicographic expression of  $a$  with a nonzero coefficient.

For  $R^{(1)} = \mathcal{O}_q(\mathbb{K}^{m \times n})$ , we will require a slight extension of Theorem 2.1.8. If  $\mathbf{t}$  is any monomial in the standard generators, then we may commute the generators in  $\mathbf{t}$  to get a lexicographic term  $\mathbf{t}^{M^{\text{lex}}}$  where  $\mathbf{t} = q^r \mathbf{t}^{M^{\text{lex}}}$  for some integer  $r$  and  $M^{\text{lex}} \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ . Since  $q^r \neq 0$ , the next result follows easily.

**Proposition 2.1.10.** *For any coordinate  $(r, s)$ , the set of lexicographic monomials of  $\mathcal{O}_q(\mathbb{K}^{m \times n})$  involving only  $t_{i,j}$  with  $(i, j) > (r, s)$  is linearly independent over the subalgebra generated by the  $t_{i,j}$  with  $(i, j) \leq (r, s)$ . Moreover, for a set  $\{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_\ell\}$  of monomials in the standard generators of  $\mathcal{O}_q(\mathbb{K}^{m \times n})$ , the following are equivalent.*

- (1) The set  $\{\mathbf{t}_1, \mathbf{t}_2, \dots, \mathbf{t}_\ell\}$  is linearly independent over  $\mathbb{K}$ .
- (2) The set  $\{\mathbf{t}_1^{M_1^{\text{lex}}}, \mathbf{t}_2^{M_2^{\text{lex}}}, \dots, \mathbf{t}_\ell^{M_\ell^{\text{lex}}}\}$  is linearly independent over  $\mathbb{K}$ .
- (3) The matrices  $M_1^{\text{lex}}, \dots, M_\ell^{\text{lex}}$  are distinct.

A similar set of statements hold for the  $m \times n$  quantum torus.  $\square$

We conclude this section by noting that  $R^{(t)}$  has a natural  $\mathbb{Z}_{\geq 0}^{m+n}$ -grading that will be quite useful in the proof of Theorem 4.4.1. If

$$\mathbf{s} = (r_1, r_2, \dots, r_m, c_1, c_2, \dots, c_n) \in (\mathbb{Z}_{\geq 0})^{m+n},$$

then the homogeneous component of degree  $\mathbf{s}$  is the subspace of  $R^{(t)}$  spanned by the lexicographic monomials of the form  $\mathbf{x}^N$ , where  $N$  satisfies

$$\begin{aligned} \sum_{j=1}^n (N)_{i,j} &= r_i, \text{ for all } i \in [m], \text{ and} \\ \sum_{i=1}^m (N)_{i,j} &= c_j, \text{ for all } j \in [n]. \end{aligned}$$

In other words, the sum of all entries in row  $i$  of  $N$  equals  $r_i$ , and the sum of all entries in column  $j$  of  $N$  equals  $c_j$ . All references in this paper to a grading on  $R^{(t)}$  will be with respect to this grading.

**2.2. The Deleting Derivations Algorithm.** The relationship between  $R^{(t)}$  and  $R^{(t-1)}$  has been studied by Cauchon [4, 5] as a special case of a more general theory. Here, we review his results as they apply to these algebras. For each result in this section, we fix  $t \in [mn]$  with  $t \neq 1$ , let  $(r, s)$  denote the  $t^{\text{th}}$  smallest coordinate, and let  $[x_{i,j}]$  be the matrix of generators of  $R^{(t)}$  and  $[y_{i,j}]$  the matrix of generators for  $R^{(t-1)}$ . All results are due to Cauchon.

**Theorem 2.2.1.**

- (1) *The multiplicative set generated by  $x_{r,s}$  is a left and right Ore set for  $R^{(t)}$ , and the multiplicative set generated by  $y_{r,s}$  is a left and right Ore set for  $R^{(t-1)}$ .*
- (2) *There is an injective homomorphism*

$$\overrightarrow{\cdot} : R^{(t-1)} \rightarrow R^{(t)} [x_{r,s}^{-1}]$$

*defined on the standard generators by*

$$\overrightarrow{y_{i,j}} = \begin{cases} x_{i,j} - x_{i,s} x_{r,s}^{-1} x_{r,j}, & \text{if } i < r \text{ and } j < s; \\ x_{i,j} & \text{otherwise.} \end{cases}$$

- (3) *There is an injective homomorphism*

$$\overleftarrow{\cdot} : R^{(t)} \rightarrow R^{(t-1)} [y_{r,s}^{-1}]$$

*defined on the standard generators by*

$$\overleftarrow{x_{i,j}} = \begin{cases} y_{i,j} + y_{i,s} y_{r,s}^{-1} y_{r,j}, & \text{if } i < r \text{ and } j < s; \\ y_{i,j} & \text{otherwise.} \end{cases}$$

$$(4) \ R^{(t)} [x_{r,s}^{-1}] = R^{(t-1)} [y_{r,s}^{-1}]. \quad \square$$

The homomorphism in Theorem 2.2.1 (2) is called the *deleting derivations map*. We call the homomorphism in Theorem 2.2.1 (3) the *adding derivations map*. (This map is called the “reverse deleting derivations map” in [14], and the “restoration” algorithm in [8].)

The strategy of Cauchon’s theory is to use these maps to iteratively transfer information between  $R^{(1)} = \mathcal{O}_q(\mathbb{K}^{m \times n})$  and  $R^{(mn)} = \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ . For example, we describe how Cauchon used this idea to embed the prime spectrum of the latter algebra into the prime spectrum of the former.

Recall that  $\text{spec}(A)$  denotes the set of prime ideals of an algebra  $A$  equipped with the usual Zariski topology. With notation as in Theorem 2.2.1, observe that we can always partition  $\text{spec}(R^{(t)})$  as

$$\text{spec}(R^{(t)}) = \text{spec}^{\not\in}(R^{(t)}) \cup \text{spec}^{\in}(R^{(t)}),$$

where

$$\text{spec}^{\not\in}(R^{(t)}) = \{P \in \text{spec}(R^{(t)}) \mid x_{r,s} \notin P\},$$

and

$$\text{spec}^{\in}(R^{(t)}) = \{P \in \text{spec}(R^{(t)}) \mid x_{r,s} \in P\}.$$

**Theorem 2.2.2.** *There exists an injective map*

$$\phi_t : \text{spec}(R^{(t)}) \rightarrow \text{spec}(R^{(t-1)})$$

*satisfying the following properties.*

- (1) *Restricted to  $\text{spec}^{\not\in}(R^{(t)})$ ,  $\phi_t$  is bijective, sending  $P \in \text{spec}^{\not\in}(R^{(t)})$  to*

$$\phi_t(P) = \overleftarrow{P}[y_{r,s}^{-1}] \cap R^{(t-1)}.$$

*If  $Q \in \text{spec}^{\not\in}(R^{(t-1)})$ , then*

$$\phi_t^{-1}(Q) = \overrightarrow{Q}[x_{r,s}^{-1}] \cap R^{(t)}.$$

- (2) *Restricted to  $\text{spec}^{\in}(R^{(t)})$ ,  $\phi_t$  is injective, sending  $P \in \text{spec}^{\in}(R^{(t)})$  to*

$$\phi_t(P) = g^{-1}(P/\langle x_{r,s} \rangle),$$

*where  $g : R^{(t-1)} \rightarrow R^{(t)}/\langle x_{r,s} \rangle$  is the unique homomorphism that maps the standard generators as  $y_{i,j} \mapsto x_{i,j} + \langle x_{r,s} \rangle$ .  $\square$*

If  $P \in \text{spec}^{\in}(R^{(t)})$ , then the next lemma provides a useful tool to identify  $\phi_t(P)$ .

**Lemma 2.2.3.** *One has  $Q \in \text{spec}^{\in}(R^{(t-1)})$  with  $Q = \phi_t(P)$  for some  $P \in \text{spec}^{\in}(R^{(t)})$  if and only if  $R^{(t)}/P \simeq R^{(t-1)}/Q$  under the homomorphism defined by  $x_{i,j} + P \mapsto y_{i,j} + Q$ .  $\square$*



**2.3.  $\mathcal{H}$ -Stratification.** For many quantum algebras, including the  $R^{(t)}$ , the structure of the prime spectrum may be understood by first understanding the prime ideals that are invariant under a rational action of an algebraic torus  $\mathcal{H}$ . For  $R^{(t)}$  with matrix of generators  $[x_{i,j}]$ , let  $\mathcal{H} = (\mathbb{K}^*)^{m+n}$  and note that every  $h = (\rho_1, \dots, \rho_m, \gamma_1, \dots, \gamma_n) \in \mathcal{H}$  induces an automorphism of  $R^{(t)}$  by

$$h \cdot x_{i,j} = \rho_i \gamma_j x_{i,j}.$$

**Definition 2.3.1.** An  $\mathcal{H}$ -prime is a prime ideal  $K \in \text{spec}(R^{(t)})$  such that  $h \cdot K = K$  for all  $h \in \mathcal{H}$ . The set of all  $\mathcal{H}$ -primes of  $R^{(t)}$  is denoted  $\mathcal{H}\text{-spec}(R^{(t)})$ . The  $\mathcal{H}$ -stratum associated to an  $\mathcal{H}$ -prime  $K$  is the set

$$\text{spec}_K(R^{(t)}) = \{P \in \text{spec}(R^{(t)}) \mid \bigcap_{h \in \mathcal{H}} h \cdot P = K\}.$$

**Theorem 2.3.2** (Goodearl-Letzter [12]). *For every  $t \in [mn]$ , there are finitely many  $\mathcal{H}$ -primes in  $\mathcal{H}\text{-spec}(R^{(t)})$ , and*

$$\text{spec}(R^{(t)}) = \bigsqcup_{K \in \mathcal{H}\text{-spec}(R^{(t)})} \text{spec}_K(R^{(t)}).$$

□

**Remark 2.3.3.** Theorem 2.2.1 and Theorem 2.3.2 are where it is necessary to require  $q$  to be a nonzero, non-root of unity. We also note here that the  $\mathcal{H}$ -primes are well-known to be homogeneous ideals.

The  $\mathcal{H}$ -primes of  $R^{(1)} = \mathcal{O}_q(\mathbb{K}^{m \times n})$  have generating sets of a simple form.

**Theorem 2.3.4** (Goodearl-Letzter [11]). *A prime ideal  $K \in \text{spec}(R^{(1)})$  is an  $\mathcal{H}$ -prime if and only if there exists a  $B \subseteq [m] \times [n]$  such that*

$$K = \langle t_{i,j} \mid (i,j) \in B \rangle.$$

□

It is convenient to describe these  $\mathcal{H}$ -primes by using diagrams.

**Definition 2.3.5.** An  $m \times n$  diagram is an  $m \times n$  grid of squares, each square colored either black or white.

We index the squares of a diagram in the usual way, i.e., as one would an  $m \times n$  matrix. If

$$K = \langle t_{i,j} \mid (i,j) \in B \rangle \in \mathcal{H}\text{-spec}(R^{(1)})$$

for some  $B \subseteq [m] \times [n]$ , then we define the diagram corresponding to  $K$  as that for which the black squares are precisely those  $(i,j) \in B$ . Conversely, any diagram defines a subset  $B \subseteq [m] \times [n]$  corresponding to the indices of the black squares, and therefore a corresponding  $K \in \mathcal{H}\text{-spec}(R^{(1)})$ . We henceforth identify a diagram with the corresponding subset  $B \subseteq [m] \times [n]$ . See Figure 1 for an example.

Fortunately, the deleting derivation map behaves nicely with respect to  $\mathcal{H}$ -primes.

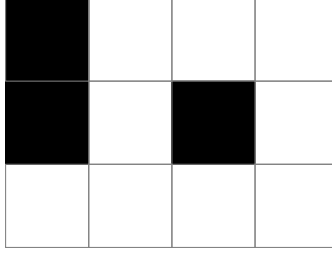


FIGURE 1. A  $3 \times 4$  diagram corresponding to the  $\mathcal{H}$ -prime  $\langle t_{1,1}, t_{2,1}, t_{2,3} \rangle \in \mathcal{H}\text{-spec}(\mathcal{O}_q(\mathbb{K}^{3 \times 4}))$ .

**Theorem 2.3.6** (Cauchon [4]). *For every  $t \in [mn]$ ,  $t \neq 1$ , the map  $\phi_t$  injects  $\mathcal{H}\text{-spec}(R^{(t)})$  into  $\mathcal{H}\text{-spec}(R^{(t-1)})$ . Consequently, the composition*

$$\phi = \phi_2 \circ \cdots \circ \phi_{mn}$$

*is an injection of  $\mathcal{H}\text{-spec}(\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K})))$  into  $\mathcal{H}\text{-spec}(\mathcal{O}_q(\mathbb{K}^{m \times n}))$ .*  $\square$

In view of the strategy mentioned in Section 2.2, a natural problem is to identify the diagrams of those  $\mathcal{H}$ -primes in  $\mathcal{H}\text{-spec}(R^{(1)})$  that are the images of  $\mathcal{H}$ -primes in  $\mathcal{H}\text{-spec}(R^{(mn)})$  under  $\phi$ .

**Definition 2.3.7.** A diagram is a *Cauchon diagram* if, for any given black square, either every square to the left or every square above is also black.

Figure 2 is an example of a Cauchon diagram, while the diagram in Figure 1 is not a Cauchon diagram since the black square in position  $(2, 3)$  has a white square above and a white square to its left.

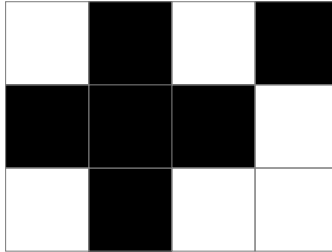


FIGURE 2. A  $3 \times 4$  Cauchon diagram.

**Theorem 2.3.8** (Cauchon [5]). *A diagram is a Cauchon diagram if and only if the corresponding  $\mathcal{H}$ -prime in  $\mathcal{H}\text{-spec}(R^{(1)})$  is the image under  $\phi$  of an  $\mathcal{H}$ -prime in  $\mathcal{H}\text{-spec}(R^{(mn)})$ .*  $\square$

### 3. QUANTUM MATRICES BY PATHS

Let  $B$  be a Cauchon diagram, and, by Theorem 2.3.8, consider the corresponding  $\mathcal{H}$ -prime  $K$  of  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ . With the notation of Section 2.3,

the image of  $K$  under the composition  $\phi_{t+1} \circ \cdots \circ \phi_{mn}$  is an  $\mathcal{H}$ -prime  $K_t$  of  $R^{(t)}$ . The goal of this section is to explain how  $R^{(t)}/K_t$  is isomorphic to a subalgebra  $A_B^{(t)}$  of the quantum torus  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$  defined by considering paths in a directed graph depending on  $B$ . In particular, when  $B = \emptyset$ , we obtain a combinatorial description of  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ .

This section is organized as follows. Section 3.1 describes the directed graph we will use to define the algebras  $A_B^{(t)}$ . We then show how to associate an element of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$  to edges and paths in this graph and conclude by discussing how the elements of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$  associated to certain paths relate. Section 3.2 defines the algebras  $A_B^{(t)}$  and Section 3.3 proves that each  $A_B^{(t)}$  is indeed isomorphic to  $R^{(t)}/K_t$ .

### 3.1. Graphs and Paths.

**Definition 3.1.1.** To a Cauchon diagram  $B$  we construct a directed graph  $G_B^{m \times n}$  called the *Cauchon graph* as follows. The vertex set consists of *white vertices*

$$W = ([m] \times [n]) \setminus B,$$

together with *row vertices*  $R = [m]$ , and *column vertices*<sup>1</sup>  $C = [n]$ . The set of directed edges  $E$  consists precisely of those in the following list.

- (1) If  $(i, j), (i, j') \in W$  are distinct white vertices with  $j > j'$  and such that there is no white vertex  $(i, j'')$  for any  $j' < j'' < j$ , then we make an edge from  $(i, j)$  to  $(i, j')$ ;
- (2) If  $(i, j), (i', j) \in W$  are distinct white vertices with  $i < i'$  such that there is no white vertex  $(i'', j)$  for any  $i < i'' < i'$ , then we make an edge from  $(i, j)$  to  $(i', j)$ ;
- (3) For  $i \in R$ , we make an edge from  $i$  to  $(i, j)$ , where  $j$  is the largest integer such that  $(i, j) \in W$  (if such a  $j$  exists);
- (4) For  $j \in C$ , we make an edge from  $(i, j)$  to  $j$  where  $i$  is the largest integer such that  $(i, j) \in W$  (if such an  $i$  exists).

There is a natural way to embed a Cauchon graph in the plane by placing it “on top” of the diagram  $B$  as follows. The white vertices are placed at the centers of the corresponding white squares, the row vertices to the right of the corresponding diagram row, and the column vertices underneath the corresponding diagram column. An example is illustrated in Figure 3. We call this the standard embedding. We will refer to aspects of a Cauchon graph using common directional terms<sup>2</sup> with the understanding that these are with respect to the standard embedding. Of fundamental importance is the next easy result which will be implicitly referred to throughout this paper.

<sup>1</sup>There is ambiguity between labels of the row and column vertices, but the type of vertex we mean will always be explicitly stated.

<sup>2</sup>For example, horizontal, vertical, above, below, northwest, etc.

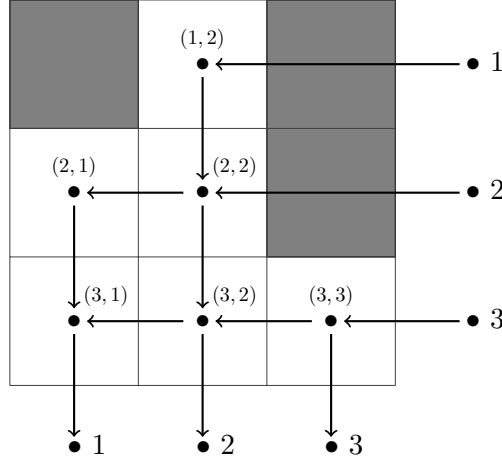


FIGURE 3. The graph  $G_B^{3 \times 3}$ , embedded on top of the  $3 \times 3$  Cauchon diagram  $B = \{(1, 1), (1, 3), (2, 3)\}$ .

**Proposition 3.1.2.** *The standard embedding of a Cauchon graph is planar, i.e., has no crossing edges.*  $\square$

We make a connection between elements of the quantum torus and Cauchon graphs by associating an element of the quantum torus to any directed path.

**Definition 3.1.3.** A *directed path* in  $G_B^{m \times n}$  is a sequence  $P = (v_0, v_1, \dots, v_k)$  of distinct vertices such that there exists an edge in  $G_B^{m \times n}$  directed from  $v_{i-1}$  to  $v_i$  for all  $i \in [k]$ . Naturally, we say that  $P$  *starts* at  $v_0$  and *ends* at  $v_k$ .

For example, in Figure 3,  $P = (1, (1, 2), (2, 2), (2, 1), (3, 1), 1)$  is a directed path starting at row vertex 1 and ending at the column vertex 1. On the other hand,  $P = (2, (2, 2), (1, 2))$  is not a directed path since there is no edge directed from  $(2, 2)$  to  $(1, 2)$ .

**Note 3.1.4.** All references to a “path” in a Cauchon graph implicitly mean a “directed path” unless otherwise stated.

**Notation 3.1.5.** If  $P$  is a path that starts at vertex  $v$  and ends at vertex  $w$ , then we write  $P: v \rightarrow w$ . If  $e$  is an edge directed from vertex  $v$  to vertex  $w$ , then we write  $e: v \rightarrow w$ . If  $e$  is an edge between two consecutive vertices in a path  $P$ , then we abuse notation by writing  $e \in P$ .

**Definition 3.1.6.** Let  $G_B^{m \times n}$  be a Cauchon graph. Define the function

$$w: E \rightarrow \mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$$

as follows, where the numbering and notation correspond to that of Definition 3.1.1:

$$(1) \ w(e: (i, j) \rightarrow (i, j')) = t_{i, j}^{-1} t_{i, j'};$$

$$(2) \ w(e: (i, j) \rightarrow (i', j)) = 1;$$

$$(3) \ w(e: i \rightarrow (i, j)) = t_{i,j};$$

$$(4) \ w(e: (i, j) \rightarrow j) = 1.$$

The value  $w(e)$  of an edge  $e$  is called the *weight* of  $e$ . If  $P = (v_0, v_1, \dots, v_k)$  is a path, and  $e_i: v_{i-1} \rightarrow v_i$ , then the weight of  $P$  is

$$w(P) = w(e_1)w(e_2) \cdots w(e_k).$$

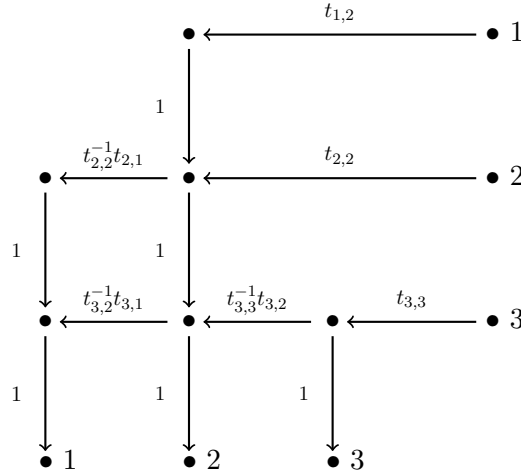


FIGURE 4. The graph  $G_B^{3 \times 3}$ , with  $B = \{(1, 1), (1, 3), (2, 3)\}$ , and edges labeled by their weights. (Labels of white vertices omitted.)

**Example 3.1.7.** Figure 4 illustrates the graph of Figure 3 with edges labeled by their weights. The weight of the path

$$P = (1, (1, 2), (2, 2), (2, 1), (3, 1), 1)$$

is

$$\begin{aligned} w(P) &= (t_{1,2})(1)(t_{2,2}^{-1}t_{2,1})(1)(1) \\ &= t_{1,2}t_{2,2}^{-1}t_{2,1}. \end{aligned}$$

It is convenient to observe that for a row vertex  $i$  and a column vertex  $j$ , the weight of a path  $P: i \rightarrow j$  can be determined by looking at the sequence of “turns”.

**Definition 3.1.8.** Let  $P = (v_0, v_1, \dots, v_{k-1}, v_k)$  be a path in a Cauchon graph starting from row vertex  $i = v_0$  and ending at column vertex  $j = v_k$ .

- A  $\Gamma$ -turn in  $P$  is a white vertex  $v_i \in P$  such that the edge from  $v_{i-1}$  to  $v_i$  is horizontal, and the edge from  $v_i$  to  $v_{i+1}$  is vertical.
- A  $\sqcup$ -turn in  $P$  is a white vertex  $v_i \in P$  such that the edge from  $v_{i-1}$  to  $v_i$  is vertical and the edge from  $v_i$  to  $v_{i+1}$  is horizontal.

**Proposition 3.1.9.** *Let  $P: i \rightarrow j$  be a path in a Cauchon graph where  $i$  is a row vertex and  $j$  is a column vertex. If  $(v_{i_1}, v_{i_2}, \dots, v_{i_t}) \subset P$  is the subsequence consisting of all  $\Gamma$ -turns and  $\mathbb{J}$ -turns, then*

$$w(P) = t_{v_{i_1}} t_{v_{i_2}}^{-1} t_{v_{i_3}} \cdots t_{v_{i_{t-1}}}^{-1} t_{v_{i_t}}.$$

*Proof.* This follows from the definitions of edge and path weights.  $\square$

**Example 3.1.10.** For the path  $P$  in Example 3.1.7, the vertex  $(1, 2)$  is a  $\Gamma$ -turn,  $(2, 2)$  is a  $\mathbb{J}$ -turn, and  $(2, 1)$  is a  $\Gamma$ -turn, so that  $w(P) = (t_{1,2})(t_{2,2}^{-1})(t_{2,1})$ . This, of course, agrees with Example 3.1.7.

Parts 1 and 2 of the next result are Lemmas 3.5 and 3.6 respectively in [3]. Part 3 is proven similarly.

**Lemma 3.1.11.** *In a Cauchon graph  $G_B^{m \times n}$ , let  $(a, b)$  be a white vertex,  $i$  and  $k$  row vertices with  $i < k$ , and  $j$  and  $\ell$  column vertices with  $j < \ell$ .*

- (1) *If  $P: i \rightarrow (a, b)$  and  $Q: (a, b) \rightarrow \ell$  are paths in  $G_B^{m \times n}$  with only  $(a, b)$  in common, then*

$$w(P)w(Q) = \begin{cases} w(Q)w(P), & \text{if } b = \ell, \text{ i.e., } Q \text{ has only vertical edges,} \\ q^{-1}w(Q)w(P), & \text{otherwise.} \end{cases}$$

- (2) *If  $P: (a, b) \rightarrow j$  and  $Q: (a, b) \rightarrow \ell$  are paths in  $G_B^{m \times n}$  with only  $(a, b)$  in common, then*

$$w(P)w(Q) = \begin{cases} w(Q)w(P), & \text{if } b = \ell, \text{ i.e., } Q \text{ has only vertical edges,} \\ qw(Q)w(P), & \text{otherwise.} \end{cases}$$

- (3) *If  $P: i \rightarrow (a, b)$  and  $Q: k \rightarrow (a, b)$  are paths in  $G_B^{m \times n}$  with only  $(a, b)$  in common, then*

$$w(P)w(Q) = qw(Q)w(P).$$

For the remainder of this section, fix  $t \in [mn]$  and let  $(r, s)$  be the  $t^{\text{th}}$  smallest coordinate.

**Notation 3.1.12.** For a row vertex  $i$  and a column vertex  $j$  of  $G_B^{m \times n}$ , let  $\Gamma_B^{(t)}(i, j)$  denote the set of all paths  $P: i \rightarrow j$  in  $G_B^{m \times n}$  for which vertex larger than  $(r, s)$  is a  $\mathbb{J}$ -turn.

Figure 5 is meant to clarify Notation 3.1.12. Note that while we have drawn a vertex  $(r, s)$  in this figure, it will not exist if  $(r, s) \in B$ . The main theorem of this section is the following. We note that the “tail-switching” technique in the proof will be used again later in this paper.

**Theorem 3.1.13.** *Let  $G_B^{m \times n}$  be a Cauchon graph and  $i \leq k$  be row vertices and  $j, \ell$  be column vertices where either  $i \neq k$  or  $j \neq \ell$ .*

- (1) *If  $i = k$  and  $j < \ell$ , then there exists a bijective function on  $\Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(i, \ell)$  sending  $(P, Q) \mapsto (\tilde{P}, \tilde{Q})$  such that*

$$w(P)w(Q) = qw(\tilde{Q})w(\tilde{P}).$$

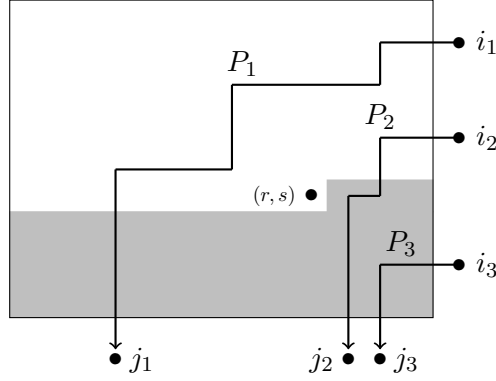


FIGURE 5. The shaded area represents all white vertices greater than  $(r, s)$ . (This convention will be repeated in later illustrations.) In this example,  $P_1 \in \Gamma_B^{(t)}(i_1, j_1)$ ,  $P_3 \in \Gamma_B^{(t)}(i_3, j_3)$  but  $P_2 \notin \Gamma_B^{(t)}(i_2, j_2)$ .

- (2) If  $i < k$  and  $j = \ell$ , then there exists a bijective function on  $\Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(k, j)$  sending  $(P, Q) \mapsto (\tilde{P}, \tilde{Q})$  such that

$$w(P)w(Q) = qw(\tilde{Q})w(\tilde{P}).$$

- (3) If  $i < k$  and  $j > \ell$ , then there exists a bijective function on  $\Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(k, \ell)$  sending  $(P, Q) \mapsto (\tilde{P}, \tilde{Q})$  such that

$$w(P)w(Q) = w(\tilde{Q})w(\tilde{P}).$$

- (4) If  $i < k$  and  $j < \ell$ , then:

- (a) If  $P \in \Gamma_B^{(t)}(i, j)$ ,  $Q \in \Gamma_B^{(t)}(k, \ell)$  and  $P \cap Q = \emptyset$ , then

$$w(P)w(Q) = w(Q)w(P);$$

- (b) There exists a bijective function from the subset of  $\Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(k, \ell)$  consisting of those  $(P, Q)$  with  $P \cap Q \neq \emptyset$ , to  $\Gamma_B^{(t)}(i, \ell) \times \Gamma_B^{(t)}(k, j)$  sending  $(P, Q)$  to  $(\tilde{P}, \tilde{Q})$  such that

$$w(P)w(Q) = qw(\tilde{Q})w(\tilde{P}).$$

*Proof. Part 1:* Let  $(P, Q) \in \Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(i, \ell)$ . Observe that  $P$  and  $Q$  have a last vertex in common, say  $(a, b)$ . See Figure 6. Therefore, we may write  $P = (P_1, P_2)$  where  $P_1: i \rightarrow (a, b)$  and  $P_2: (a, b) \rightarrow j$ , and  $Q = (Q_1, Q_2)$  where  $Q_1: k \rightarrow (a, b)$  and  $Q_2: (a, b) \rightarrow \ell$ . Define  $\tilde{P} = (Q_1, P_2)$  and  $\tilde{Q} = (P_1, Q_2)$ . Notice that  $\tilde{P}$  is a path from  $i$  to  $j$  and that  $\tilde{\tilde{P}} = P$ , and  $\tilde{Q}$  is a path from  $i$  to  $\ell$  and  $\tilde{\tilde{Q}} = Q$ . Now  $P_2$  necessarily has a horizontal edge. If

$Q_2$  has only vertical edges, then

$$\begin{aligned}
 w(P)w(Q) &= w(P_1)w(P_2)w(Q_1)w(Q_2) \\
 &= qw(P_1)w(Q_1)w(P_2)w(Q_2) \text{ (Lemma 3.1.11, Part 1),} \\
 &= qw(P_1)w(Q_2)w(Q_1)w(P_2) \text{ (Lemma 3.1.11, Parts 1\& 3,)} \\
 &= qw(\tilde{Q})w(\tilde{P}).
 \end{aligned}$$

If  $Q_2$  has a horizontal edge, then

$$\begin{aligned}
 w(P)w(Q) &= w(P_1)w(P_2)w(Q_1)w(Q_2) \\
 &= q^{-1}qw(P_1)w(Q_2)w(P_2)w(Q_1) \text{ (Lemma 3.1.11, Part 1).} \\
 &= qw(P_1)w(Q_2)w(Q_1)w(P_2) \text{ (Lemma 3.1.11, Part 1\& 3,)} \\
 &= qw(\tilde{Q})w(\tilde{P}).
 \end{aligned}$$

Finally, by the construction of  $(\tilde{P}, \tilde{Q})$  from  $(P, Q)$ , the multiset of edges used by  $P$  and  $Q$  equals the multiset of edges used by  $\tilde{P}$  and  $\tilde{Q}$  so that  $(\tilde{P}, \tilde{Q}) \in \Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(i, \ell)$  and the map sending  $(P, Q)$  to  $(\tilde{P}, \tilde{Q})$  is an involution and so a bijection.

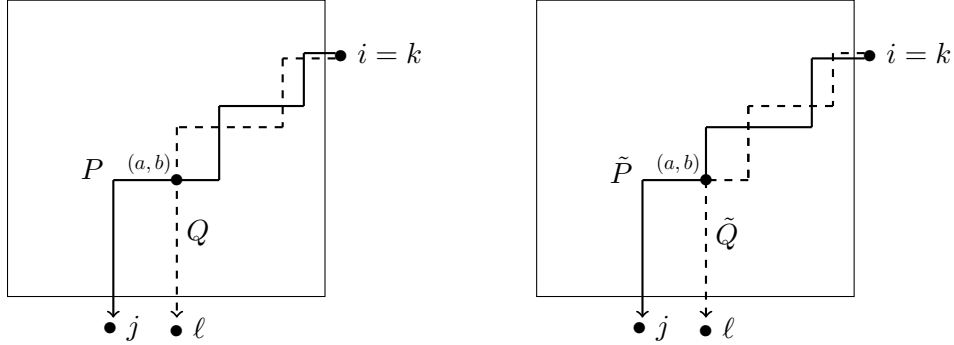


FIGURE 6. Illustration of Part 1 in the proof of Theorem 3.1.13. The left figure shows paths  $P$  (solid) and  $Q$  (dashed). Right figure shows paths  $\tilde{P}$  (solid) and  $\tilde{Q}$  (dashed).

*Part 2:* Let  $(P, Q) \in \Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(k, j)$ . In this case,  $P$  and  $Q$  have a first common vertex, say  $(a, b)$ . Therefore, we may write  $P = (P_1, P_2)$  where  $P_1: i \rightarrow (a, b)$  and  $P_2: (a, b) \rightarrow j$ , and  $Q = (Q_1, Q_2)$  where  $Q_1: k \rightarrow (a, b)$  and  $Q_2: (a, b) \rightarrow \ell$ . Define  $\tilde{P} = (P_1, Q_2)$  and  $\tilde{Q} = (Q_1, P_2)$ . The remainder of the proof of this part proceeds as in Part 1 using Lemma 3.1.11, Parts 1 and 2.

*Part 3:* In this case,  $P$  and  $Q$  have a first common vertex  $(a, b)$  and a last common vertex  $(a', b')$ . We can write  $P = (P_1, P_2, P_3)$  where  $P_1: i \rightarrow (a, b)$ ,  $P_2: (a, b) \rightarrow (a', b')$  and  $P_3: (a', b') \rightarrow j$ . Similarly  $Q = (Q_1, Q_2, Q_3)$  where  $Q_1: k \rightarrow (a, b)$ ,  $Q_2: (a, b) \rightarrow (a', b')$  and  $Q_3: (a', b') \rightarrow \ell$ . Define  $\tilde{P} =$



$(P_1, Q_2, P_3)$  and  $\tilde{Q} = (Q_1, P_2, Q_3)$ . There are several possibilities obtained by considering whether or not  $P_2, P_3$  and  $Q_2$  consist only of vertical edges (or no edges at all), but in every case, the calculations will give  $w(P)w(Q) = w(\tilde{Q})w(\tilde{P})$  as well as the additional statements in Part 3.

*Part 4a:* First, suppose  $P$  turns at vertex  $(a, b)$  where  $a \neq k$  and  $b \neq \ell$ . By the commutation relations for  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ ,  $t_{a,b}^{\pm 1}$  commutes with all generators  $t_{c,d}^{\pm 1}$  appearing in  $w(Q)$  except those with  $c = a$  or  $d = b$ . However, since  $a \neq k$ ,  $Q$  J-turns at a vertex  $(a, d)$  if and only if  $Q$   $\Gamma$ -turns at a vertex  $(a, d')$  for some  $a < d' < d$ . On the other hand,  $t_{c,d}^{\pm 1}$  commutes with  $t_{a,d}^{-1}t_{a,d'}$ . Also, since  $b \neq \ell$ ,  $Q$   $\Gamma$ -turns at a vertex  $(c, b)$  if and only if  $Q$  J-turns at a vertex  $(c', b)$  for some  $b < c < c'$ , and  $t_{c,d}^{\pm 1}$  commutes with  $t_{c,b}t_{c',b}^{-1}$ .

Next, suppose  $(k, d)$  is the first vertex at which  $Q$   $\Gamma$ -turns and  $(c, \ell)$  is the last vertex at which  $Q$   $\Gamma$ -turns. Notice that  $P$  J-turns at vertex  $(k, b)$  if and only if  $P$   $\Gamma$ -turns at a vertex  $(k, b')$  where  $b' < b < d$  and that  $t_{k,b}^{-1}t_{k,b'}$  commutes with  $t_{k,d}$ . Similarly,  $P$   $\Gamma$ -turns at vertex  $(a, \ell)$  if and only if  $P$  J-turns at vertex  $(a', \ell)$  for some  $a < a' < c$  and  $t_{a,\ell}^{-1}t_{a',\ell}$  commutes with  $t_{c,\ell}$ . These two facts combined with the previous paragraph shows imply that  $w(P)$  commutes with  $w(Q)$ , as desired.

*Part 4b:* As in Part 1, we let  $(a, b)$  be the last common vertex in a non-disjoint pair of paths  $(P, Q) \in \Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(k, \ell)$ . We then “switch” the tails of  $P$  and  $Q$  at  $(a, b)$  to obtain  $\tilde{P}: i \rightarrow \ell$  and  $\tilde{Q}: k \rightarrow j$ . The remainder of the proof is as in Part 1.  $\square$

**3.2. The Algebras  $A_B^{(t)}$ .** In this section we introduce, for each  $t \in [mn]$  and Cauchon diagram  $B$ , a subalgebra  $A_B^{(t)}$  of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ . When  $B = \emptyset$ , we will see that  $A_\emptyset^{(t)} \simeq R^{(t)}$ . Throughout this section we fix  $t \in [mn]$  and let  $(r, s)$  be the  $t^{\text{th}}$  smallest coordinate.

**Definition 3.2.1.** We define  $A_B^{(t)}$  to be the subalgebra of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$  with the  $m \times n$  matrix of generators  $[x_{i,j}]$  where, for each coordinate  $(i, j)$ ,

$$x_{i,j} = \sum_{P \in \Gamma_B^{(t)}(i,j)} w(P).$$

When  $B = \emptyset$  we write  $A^{(t)} = A_\emptyset^{(t)}$ .

**Example 3.2.2.** Consider the  $2 \times 3$  Cauchon diagram  $B = \{(1, 1)\}$ . Figure 7 presents two copies of the corresponding Cauchon graph, where we continue the illustrative convention that no path may contain a J-turn in the shaded region. For each  $t \in [6]$ , we denote by  $[x_{i,j}^{(t)}]$  the matrix of generators for  $A_B^{(t)}$ .

The left graph of Figure 7 corresponds to  $t = 1$ . In this case, any path from row vertex 1 to column vertex 1 necessarily contains a J-turn in the

shaded region. Therefore,  $A_B^{(1)}$  has the matrix of generators

$$\begin{bmatrix} x_{1,1}^{(1)} & x_{1,2}^{(1)} & x_{1,3}^{(1)} \\ x_{2,1}^{(1)} & x_{2,2}^{(1)} & x_{2,3}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \end{bmatrix}.$$

One may check that  $A_B^{(1)} = A_B^{(2)} = A_B^{(3)} = A_B^{(4)}$ . For  $t = 5$ , the Cauchon graph is illustrated on the right in Figure 7. In this case, there exists a unique path in  $\Gamma_B^{(5)}(1, 1)$ , so that the matrix of generators for  $A_B^{(5)}$  is

$$\begin{bmatrix} x_{1,1}^{(5)} & x_{1,2}^{(5)} & x_{1,3}^{(5)} \\ x_{2,1}^{(5)} & x_{2,2}^{(5)} & x_{2,3}^{(5)} \end{bmatrix} = \begin{bmatrix} t_{1,2}t_{2,2}^{-1}t_{2,1} & t_{1,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \end{bmatrix}.$$

Finally, one may check that  $A_B^{(6)}$  has matrix of generators

$$\begin{bmatrix} x_{1,1}^{(6)} & x_{1,2}^{(6)} & x_{1,3}^{(6)} \\ x_{2,1}^{(6)} & x_{2,2}^{(6)} & x_{2,3}^{(6)} \end{bmatrix} = \begin{bmatrix} t_{1,2}t_{2,2}^{-1}t_{2,1} + t_{1,3}t_{2,3}^{-1}t_{2,1} & t_{1,2} + t_{1,3}t_{2,3}^{-1}t_{2,2} & t_{1,3} \\ t_{2,1} & t_{2,2} & t_{2,3} \end{bmatrix}.$$

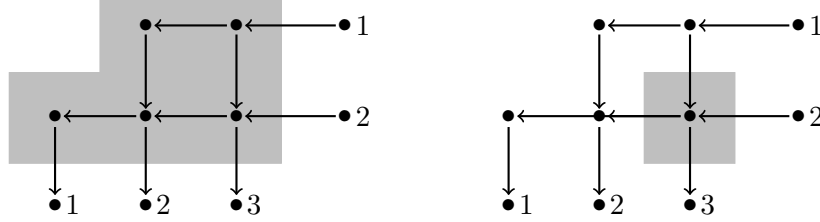


FIGURE 7. Two copies of the graph  $G_{\{(1,1)\}}^{2 \times 3}$  referred to in Example 3.2.2. The left picture is shaded to assist the definition of  $A_B^{(1)}$ , the right picture for  $A_B^{(5)}$ .

Theorem 3.1.13 implies some commutation relations between the generators of  $A_B^{(t)}$ .

**Theorem 3.2.3** (cf. Definition 2.1.4). *If  $X = [x_{i,j}]$  is the matrix of generators for  $A_B^{(t)}$ , and*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

*is any  $2 \times 2$  submatrix of  $X$ , then:*

- (1)  $ab = qba$ ,  $cd = qdc$ ;
- (2)  $ac = qca$ ,  $bd = qdb$ ;
- (3)  $bc = cb$ ;
- (4)  $ad = \begin{cases} da, & \text{if } d = x_{k,\ell} \text{ and } (k, \ell) > (r, s); \\ da + (q - q^{-1})bc, & \text{if } d = x_{k,\ell} \text{ and } (k, \ell) \leq (r, s). \end{cases}$

*Proof.* First note that for any coordinates  $(i, j)$  and  $(i', j')$ ,

$$\begin{aligned} x_{i,j}x_{i',j'} &= \sum_{\substack{P \in \Gamma_B^{(t)}(i,j), \\ Q \in \Gamma_B^{(t)}(i',j')}} w(P)w(Q) \\ &= \sum_{\substack{P, Q: \\ P \cap Q = \emptyset}} w(P)w(Q) + \sum_{\substack{P, Q: \\ P \cap Q \neq \emptyset}} w(P)w(Q). \end{aligned} \quad (1)$$

Let

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} x_{i,j} & x_{i,\ell} \\ x_{k,j} & x_{k,\ell} \end{bmatrix}$$

be a  $2 \times 2$  submatrix of  $X$ .

First, consider  $x_{i,j}$  and  $x_{i,\ell}$ . In this case the first sum in Equation (1) is necessarily empty, since any pair  $(P, Q) \in \Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(i, \ell)$  have row vertex  $i$  in common. Part 1 of Theorem 3.1.13 shows that for any such pair, there is a unique pair  $(\tilde{P}, \tilde{Q}) \in \Gamma_B^{(t)}(i, j) \times \Gamma_B^{(t)}(i, \ell)$  such that  $w(P)w(Q) = qw(\tilde{Q})w(\tilde{P})$ . Hence, Equation (1) implies  $x_{i,j}x_{i,\ell} = qx_{i,\ell}x_{i,j}$ . The relations between:  $x_{k,j}$  and  $x_{k,\ell}$ ;  $x_{i,j}$  and  $x_{k,j}$ ;  $x_{i,\ell}$  and  $x_{k,\ell}$ ; and  $x_{i,j}$  and  $x_{k,j}$  are all obtained similarly.

Now consider  $x_{i,j}$  and  $x_{k,\ell}$ . If  $(r, s) < (k, \ell)$ , then

$$\Gamma_B^{(t)}(k, \ell) = \{Q = (k, (k, \ell), \ell)\}$$

and any  $P \in \Gamma_B^{(t)}(i, j)$  is disjoint from  $Q$  by definition of  $\Gamma_B^{(t)}(i, j)$ . Hence  $x_{i,j}x_{k,\ell} = x_{k,\ell}x_{i,j}$  by Part 4a of Theorem 3.1.13. If  $(k, \ell) \leq (r, s)$ , then by Equation (1) and Part 4b of Theorem 3.1.13, we obtain

$$x_{i,j}x_{k,\ell} = qx_{i,\ell}x_{k,j} + \sum_{\substack{P \in \Gamma_B^{(t)}(i,j), Q \in \Gamma_B^{(t)}(i,j): \\ P \cap Q = \emptyset}} w(P)w(Q).$$

Since the weights of disjoint paths commute by Part 4a of Theorem 3.1.13, it follows that  $x_{i,j}x_{k,\ell} - x_{k,\ell}x_{i,j} = (q - q^{-1})x_{i,\ell}x_{k,j}$ .  $\square$

The intuition behind these algebras is that one obtains  $A_B^{(t)}$  from  $A_B^{(t-1)}$  by “allowing more paths.” To be more precise, let  $[x_{i,j}]$  be the matrix of generators for  $A_B^{(t)}$ , and  $[y_{i,j}]$  that of  $A_B^{(t-1)}$ . Notice that, as elements of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ ,

$$x_{i,j} = y_{i,j} + \sum w(P), \quad (2)$$

where the sum is over all paths  $P: i \rightarrow j$  for which  $(r, s)$  is a  $\mathbb{J}$ -turn in  $P$ . If  $i \geq r$ ,  $j \geq s$ , or  $(r, s) \in B$ , then no such  $P$  exists and

$$x_{i,j} = y_{i,j}.$$

On the other hand, if  $(r, s) \notin B$  and both  $i < r$  and  $j < s$ , suppose  $P: i \rightarrow j$  is a path with a  $\mathbb{J}$ -turn at  $(r, s)$ . Consider  $w(P)w(Q)$ , where  $Q = (r, (r, s), s)$ .

As in the proof of Theorem 3.1.13, we may form paths  $\tilde{P}: i \rightarrow s$  and  $\tilde{Q}: r \rightarrow j$  by “switching tails” at  $(r, s)$ . Since  $w(P)w(Q) = qw(\tilde{Q})w(\tilde{P})$ , multiplying Equation (2) through by  $y_{r,s} = x_{r,s} = w(Q)$  gives

$$\begin{aligned} x_{i,j}x_{r,s} &= y_{i,j}y_{r,s} + \sum w(P)y_{r,s} \\ &= y_{i,j}y_{r,s} + qy_{i,s}y_{r,j}. \end{aligned}$$

One may easily check that  $t_{r,s} = x_{r,s} = y_{r,s}$  generates a left and right Ore set for  $A_B^{(t)}$  and  $A_B^{(t-1)}$ . (For  $x_{r,s}$ , this follows from the observation that  $x_{i,j}x_{r,s}^{m+1} = x_{r,s}^m a$  for some  $a \in A_B^{(t)}$  when  $x_{i,j} \neq 0$  and  $(i, j)$  is northwest of  $(r, s)$ .) Hence, we have just proved Parts 1 and 2 of the following result. Part 3 follows from these, and Part 4 is trivial.

**Theorem 3.2.4** (cf. Proposition 5.4.2 in [4]). *The following hold.*

- (1) *If  $(r, s) \notin B$ , then  $A_B^{(t-1)}$  is a subalgebra of*

$$A_B^{(t)}[x_{r,s}^{-1}]$$

*where*

$$y_{i,j} = \begin{cases} x_{i,j} - x_{i,s}(x_{r,s})^{-1}x_{r,j}, & \text{if } i < r \text{ and } j < s; \\ x_{i,j} & \text{otherwise.} \end{cases}$$

- (2) *If  $(r, s) \notin B$ , then  $A_B^{(t)}$  is a subalgebra of*

$$A_B^{(t-1)}[y_{r,s}^{-1}]$$

*where*

$$x_{i,j} = \begin{cases} y_{i,j} + y_{i,s}(y_{r,s})^{-1}y_{r,j}, & \text{if } i < r \text{ and } j < s; \\ y_{i,j} & \text{otherwise.} \end{cases}$$

- (3) *If  $(r, s) \notin B$ , then  $A_B^{(t)}[x_{r,s}^{-1}] = A_B^{(t-1)}[y_{r,s}^{-1}]$ .*

- (4) *If  $(r, s) \in B$ , then  $A_B^{(t)} = A_B^{(t-1)}$ .* □

In view of Theorem 2.2.1, we may conclude the following when  $B = \emptyset$ .

**Corollary 3.2.5.** *For every  $t \in [mn]$  we have  $R^{(t)} \simeq A^{(t)}$ , where  $R^{(t)}$  are the algebras of Definition 2.1.4, and where the standard generator of  $R^{(t)}$  with coordinate  $(i, j)$  maps to the generator of  $A^{(t)}$  with coordinate  $(i, j)$ .*

Hence,  $A^{(1)} \simeq \mathcal{O}_q(\mathbb{K}^{m \times n})$ ,  $A^{(mn)} \simeq \mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$  and both the deleting derivations and  $\mathcal{H}$ -stratification theories apply to  $A^{(t)}$ . Moreover, we follow the arrow notation introduced in Section 2.2 to distinguish a generator  $x_{i,j}$  of  $A_B^{(t)}$  from its image  $\overleftarrow{x}_{i,j}$  in  $A_B^{(t-1)}$ , and a generator  $y_{i,j}$  of  $A_B^{(t-1)}[y_{r,s}^{-1}]$  from its image  $\overrightarrow{y}_{i,j}$  in  $A_B^{(t)}[x_{r,s}^{-1}]$ .

**3.3.  $\mathcal{H}$ -Primes as Kernels.** Throughout this section we fix  $t \in [mn]$  and a Cauchon diagram  $B$ . Denote the matrix of generators for  $A^{(t)}$  by  $[x_{i,j}]$  and the matrix of generators for  $A_B^{(t)}$  by  $[x_{i,j}^B]$ .

**Definition 3.3.1.** For  $t \in [mn]$  and a Cauchon diagram  $B$ , let  $\sigma_B^{(t)} : A^{(t)} \rightarrow A_B^{(t)}$  be defined on the standard generators by

$$\sigma_B^{(t)}(x_{i,j}) = x_{i,j}^B.$$

**Proposition 3.3.2.** *The map  $\sigma_B^{(t)}$  extends to a well-defined, surjective homomorphism.*

*Proof.* Recall that  $A^{(t)}$  has a presentation by generators  $x_{i,j}$  and relations as per Theorem 3.2.3. Since this theorem also tells us that the corresponding generators  $x_{i,j}^B$  of  $A_B^{(t)}$  satisfy the same relations, it follows by the universal property of presentations that  $\sigma_B^{(t)}$  is a well-defined homomorphism. Surjectivity is then clear from the definition of  $\sigma_B^{(t)}$ .  $\square$

The next result explains our interest in the  $A_B^{(t)}$  rather than just  $A^{(t)}$ .

**Theorem 3.3.3.** *One has*

$$\ker(\sigma_B^{(t)}) \in \mathcal{H}\text{-spec}(A^{(t)}).$$

Moreover, if  $t > 1$ ,

$$\ker(\sigma_B^{(t-1)}) = \phi_t(\ker(\sigma_B^{(t)})),$$

where  $\phi_t$  is as in Theorem 2.2.2.

This theorem follows as a special case of Lemmas 5.3.1 and 5.3.2 in [4]. However, as it is fundamental to the remainder of this paper, we include a proof here for completeness.

*Proof.* By Theorem 2.3.8, there is a sequence of  $\mathcal{H}$ -primes

$$(J_1, J_2, \dots, J_{mn})$$

where  $J_1$  is the  $\mathcal{H}$ -prime of  $R^{(1)} \simeq A^{(1)}$  corresponding to  $B$ , and

$$J_{t-1} = \phi_t(J_t)$$

for every  $t > 1$ . Let

$$K_t = \ker(\sigma_B^{(t)}).$$

We proceed by induction on  $t$ .

When  $t = 1$ ,

$$K_1 \supseteq J_1 = \langle t_{i,j} \mid (i,j) \in B \rangle,$$

since  $\sigma_B^{(t)}(t_{i,j}) = 0$  if and only if  $(i,j) \in B$ .

For  $a \in K_1$ , suppose  $a$  has a lexicographic expression

$$a = \sum_{M \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})} \alpha_M \mathbf{t}^M.$$

As the matrices  $M$  for which  $\mathbf{t}^M$  has a nonzero coefficient are distinct, we see that

$$0 = \sigma_B^{(1)}(a) = \sum_M \alpha_M \sigma_B^{(1)}(\mathbf{t}^M) \quad (3)$$

is a linear combination of lexicographic terms in  $\mathcal{O}_q(\mathbb{K}^{m \times n})$ . Since  $\alpha_M \neq 0$ , Equation (3) holds if and only if each  $\sigma_B^{(1)}(\mathbf{t}^M) = 0$  by Proposition 2.1.10. Since  $\mathcal{O}_q(\mathbb{K}^{m \times n})$  is a domain,  $\sigma_B^{(1)}(\mathbf{t}^M) = 0$  if and only if there is an  $(i, j) \in B$  with  $(M)_{i,j} \geq 1$ . Thus,  $K_1 \subseteq J_1$ .

Suppose that  $t > 1$  and  $K_{t-1} = J_{t-1}$ . Denote the matrix of generators for  $A^{(t-1)}$  (respectively  $A_B^{(t-1)}$ ) by  $[y_{i,j}]$  (respectively  $[y_{i,j}^B]$ ). Let  $(r, s)$  denote the  $t^{\text{th}}$  smallest coordinate. There are two cases to consider according to whether or not  $(r, s) \in B$ .

If  $(r, s) \in B$ , then for each coordinate  $(i, j)$ ,  $y_{i,j}^B = x_{i,j}^B$  as elements of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ , so that

$$A^{(t)}/K_t \simeq A_B^{(t)} = A_B^{(t-1)} \simeq A^{(t-1)}/K_{t-1}.$$

Since  $x_{r,s}^B = y_{r,s}^B = 0$ , we have  $x_{r,s} \in K_t$  and  $y_{r,s} \in K_{t-1}$ , so that we may apply Lemma 2.2.3 and induction to see

$$\phi_t(K_t) = K_{t-1} = J_{t-1} = \phi_t(J_t).$$

Since  $\phi_t$  is injective,  $K_t = J_t$ .

Next, suppose  $(r, s) \notin B$ . Consider the map

$$\tilde{\sigma}_B^{(t-1)} : A^{(t-1)}[y_{r,s}^{-1}] \rightarrow A_B^{(t-1)}[(y_{r,s}^B)^{-1}]$$

defined by

$$\tilde{\sigma}_B^{(t-1)}(ay_{r,s}^{-h}) = \sigma_B^{(t-1)}(a)(y_{r,s}^B)^{-h},$$

and the map

$$\tilde{\sigma}_B^{(t)} : A^{(t)}[x_{r,s}^{-1}] \rightarrow A_B^{(t)}[(x_{r,s}^B)^{-1}]$$

defined by

$$\tilde{\sigma}_B^{(t)}(ax_{r,s}^{-h}) = \sigma_B^{(t)}(a)(x_{r,s}^B)^{-h}.$$

These maps are well-defined homomorphisms by reasons similar to the proof of Proposition 3.3.2.

$$\begin{array}{ccc}
A^{(t-1)}[y_{r,s}^{-1}] & \xrightarrow{\tilde{\sigma}_B^{(t-1)}} & A_B^{(t-1)}[(y_{r,s}^B)^{-1}] \\
\downarrow \wr & & \downarrow \wr \\
A^{(t)}[x_{r,s}^{-1}] & \xrightarrow{\tilde{\sigma}_B^{(t)}} & A_B^{(t)}[(x_{r,s}^B)^{-1}]
\end{array}$$

FIGURE 8

It is easy to check that the diagram in Figure 8 commutes (where the isomorphisms are the adding derivation maps) so that

$$\begin{aligned}
K_t[x_{r,s}^{-1}] &= \ker \left( \tilde{\sigma}^{B,(t)} \right) \\
&= \overrightarrow{\ker \left( \tilde{\sigma}^{B,(t-1)} \right)} \\
&= \overrightarrow{K_{t-1}}[x_{r,s}^{-1}].
\end{aligned}$$

Since  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$  is a domain, so is  $A_B^{(t)}$ , and hence  $K_t$  is a prime ideal. Applying Cauchon's map, induction, and Cauchon's map again we see

$$K_t = \overrightarrow{K_{t-1}} \cap A^{(t)} = \overrightarrow{J_{t-1}} \cap A^{(t)} = J_t.$$

□

For  $M \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ , write  $M = M_0 + M_1$ , where

$$(M_0)_{i,j} = \begin{cases} (M)_{i,j} & \text{if } (i,j) \leq (r,s); \\ 0 & \text{if } (i,j) > (r,s), \end{cases}$$

and  $M_1 = M - M_0$ . Now, with notation as in the proof of Theorem 3.3.3, fix  $a \in K_t$ . Let  $\mathcal{M}$  denote the set of  $M \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$  for which  $\mathbf{x}^M$  appears in  $a$ . Hence, for some  $\alpha_M \in \mathbb{K}^*$ , we have

$$\begin{aligned}
a &= \sum_{M \in \mathcal{M}} \alpha_M \mathbf{x}^M \\
&= \sum_{M \in \mathcal{M}} \alpha_M \mathbf{x}^{M_0} \mathbf{x}^{M_1} \\
&= \sum_{N \in \mathcal{M}_{m,n}(\mathbb{Z})} \left( \sum_{\substack{M \in \mathcal{M}: \\ M_1 = N_1}} \alpha_M \mathbf{x}^{M_0} \right) \mathbf{x}^{N_1}.
\end{aligned}$$

Consider

$$\sigma_B^{(t)}(a) = \sum_{N \in \mathcal{M}_{m,n}(\mathbb{Z})} \left( \sum_{\substack{M \in \mathcal{M}: \\ M_1 = N_1}} \alpha_M \sigma_B^{(t)}(\mathbf{x}^{M_0}) \right) \sigma_B^{(t)}(\mathbf{x}^{N_1}) = 0 \quad (4)$$

Let  $N \in \mathcal{M}_{m,n}(\mathbb{Z})$ . If there is a coordinate  $(i, j) > (r, s)$  with both  $(i, j) \in B$  and  $(N)_{i,j} \geq 1$ , then  $\mathbf{x}^{N_1} \in K_t$  since  $x_{i,j} = t_{i,j}$  and  $\sigma_B^{(t)}(x_{i,j}) = 0$ . Otherwise,  $\mathbf{x}^{N_1} \neq 0$ , and the coefficient of  $\mathbf{x}^{N_1}$  in Equation (4) is an element of the subalgebra of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$  generated by those  $t_{i,j}$  with  $(i, j) \leq (r, s)$ . By Proposition 2.1.10, it follows that this coefficient is, in fact, zero, i.e., that

$$\sum_{\substack{M \in \mathcal{M}: \\ M_1 = N_1}} \alpha_M \mathbf{x}^{M_0} \in K_t.$$

The next lemma summarizes the preceding two paragraphs.

**Lemma 3.3.4.** *If  $a \in K_t$ , then*

$$a = a' + \sum_{\substack{N \in \mathcal{M}_{m,n}(\mathbb{Z}), \\ \mathbf{x}^{N_1} \notin K_t}} a_N \mathbf{x}^{N_1},$$

where each  $a_N \in K_t$  and  $a' \in K_t$  has the property that every lexicographic term  $\mathbf{x}^N$  appearing in  $a'$  satisfies  $\mathbf{x}^{N_1} \in K_t$ , i.e.,  $(N)_{i,j} \geq 1$  for some  $(i, j) > (r, s)$  and  $(i, j) \in B$ .

#### 4. GENERATORS OF $\mathcal{H}$ -PRIMES

The goal of this section is the proof of Theorem 4.4.1 where we show that an  $\mathcal{H}$ -prime in  $\mathcal{H}\text{-spec}(\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K})))$  has, as a left ideal, a Gröbner basis consisting of the quantum minors it contains. That these elements also form a Gröbner basis as a right ideal can be shown similarly.

We begin by defining quantum minors in Section 4.1 and recalling results of [3] that, in the cases of interest to us, show these elements have a particularly nice form in terms of the paths viewpoint via a  $q$ -analogue of Lindström's classic lemma [16]. We follow this by recalling the basic notions of Gröbner bases as applied to the algebras  $A^{(t)}$ , and finally prove the main result.

**4.1. Quantum Minors.** Throughout this section, we fix a Cauchon diagram  $B$ ,  $t \in [mn]$ , let  $(r, s)$  be the  $t^{\text{th}}$  smallest coordinate, and let  $[x_{i,j}]$  be the matrix of generators for  $A_B^{(t)}$

**Definition 4.1.1.** Let  $I = \{i_1 < i_2 < \dots < i_k\} \subseteq [m]$  and  $J = \{j_1 < j_2 < \dots < j_k\} \subseteq [n]$  be nonempty subsets of the same cardinality. The *quantum minor* associated to  $I$  and  $J$  is the element of  $A_B^{(t)}$  defined by

$$[I | J]_B^{(t)} = \sum_{\sigma \in S_k} (-q)^{\ell(\sigma)} x_{i_1, j_{\sigma(1)}} \cdots x_{i_k, j_{\sigma(k)}}$$

where  $S_k$  is the set of permutations of  $[k]$  and  $\ell(\sigma)$  is the number of inversions of  $\sigma \in S_k$ , i.e., the number of pairs  $i, i' \in [k]$  with  $i < i'$  but  $\sigma(i) > \sigma(i')$ .



**Remark 4.1.2.** The defining expression for  $[I | J]_B^{(t)}$  is its lexicographic expression. More precisely, for  $\sigma \in S_k$ , write  $P_\sigma$  to be the  $m \times n$  matrix whose submatrix indexed by  $(I, J)$  equals the standard  $k \times k$  permutation matrix corresponding to  $\sigma$ , and where all other entries of  $P_\sigma$  are zero. We can then write

$$[I | J]_B^{(t)} = \sum_{\sigma \in S_k} (-q)^{\ell(\sigma)} \mathbf{x}^{P_\sigma}.$$

For the remainder of this section, we write  $[I | J] = [I | J]_B^{(t)}$ . For the remainder of this paper will often write “minor” to mean “quantum minor”.

**Definition 4.1.3.** For  $I = \{i_1 < i_2 < \dots < i_k\} \subseteq [m]$  and  $J = \{j_1 < j_2 < \dots < j_k\} \subseteq [n]$ , each  $(i_\ell, j_\ell)$  is called a *diagonal coordinate* of  $[I | J]$ . Moreover,  $(i_k, j_k)$  is the *maximum coordinate* of  $[I | J]$ .

As elements of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ , each minor whose maximum coordinate is at most  $(r, s)$  reduces to a particularly nice form via a  $q$ -analogue of Lindström’s Lemma. To explain, we first need to set some notation.

**Definition 4.1.4.** Recall the notions and notation defined in Section 3.1. Let  $I = \{i_1, \dots, i_k\} \subseteq [m]$  and  $J = \{j_1, \dots, j_k\} \subseteq [n]$  be such that  $|I| = |J| = k$ .

- (1) A *vertex-disjoint path system* from the row vertices  $I$  to the column vertices  $J$  in  $G_B^{m \times n}$  is a set of  $k$  mutually disjoint paths  $(P_1, \dots, P_k)$  where  $P_r \in \Gamma_B^{(t)}(i_r, j_r)$  for each  $r \in [k]$ . We write

$$\Gamma_B^{(t)}(I | J) = \{\text{all vertex-disjoint path systems from } I \text{ to } J \text{ in } G_B^{m \times n}\}.$$

- (2) If  $\mathcal{P} = (P_1, \dots, P_k) \in \Gamma_B^{(t)}(I | J)$ , then the *weight* of  $\mathcal{P}$  is the product  $w(\mathcal{P}) = w(P_1)w(P_2) \cdots w(P_k) \in \mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ .

**Notation 4.1.5.** If we wish to explicitly write out the elements of  $I$  and  $J$  in either  $[I | J]$  or  $\Gamma_B^{(t)}(I | J)$ , we will omit the braces. For example, we write

$$[I | J] = [\{i_1, \dots, i_k\} | \{j_1, \dots, j_k\}] = [i_1, \dots, i_k | j_1, \dots, j_k].$$

**Example 4.1.6.** For the Cauchon graph of Figure 9, the path system  $\mathcal{P} = (P_1, P_2, P_3)$  where

$$\begin{aligned} P_1 &= (1, (1, 3), (1, 2), (2, 2), (4, 2), (4, 1), 1), \\ P_2 &= (2, (2, 3), (3, 3), (4, 3), 3), \\ P_3 &= (4, (4, 4), 4) \end{aligned}$$

is a vertex-disjoint path system in  $\Gamma_B^{(16)}(1, 2, 3 | 1, 3, 4)$ . In fact, it is the unique such vertex-disjoint path system and

$$w(\mathcal{P}) = (t_{1,2}t_{4,2}^{-1}t_{4,1})(t_{2,3})(t_{4,4}).$$

The reader may verify that the set  $\Gamma_B^{(16)}(1, 2 | 1, 2)$  is empty.



Similarly, if one so wishes, it may be checked that

$$\begin{aligned}
[1, 2, 3 \mid 1, 3, 4] &= x_{1,1}x_{2,3}x_{3,4} - qx_{1,1}x_{2,4}x_{3,3} - qx_{1,3}x_{2,1}x_{3,4} - q^3x_{1,4}x_{2,3}x_{3,1} \\
&\quad + q^2x_{1,3}x_{2,4}x_{3,1} + q^2x_{1,4}x_{2,1}x_{3,3} \\
&= w(P_1)w(P_2)w(P_3) \\
&= (t_{1,2}t_{4,2}^{-1}t_{4,1})(t_{2,3})(t_{3,4}),
\end{aligned}$$

where  $P_1, P_2$  and  $P_3$  are as in Example 4.1.6.

Clearly, Theorem 4.1.7 is at the very least a useful time saving device. Let us mention one quick application: the well-known fact that in  $\mathcal{O}_q(\mathcal{M}_{n,n}(\mathbb{K}))$  the *quantum determinant*

$$D_q = [1, 2, \dots, n \mid 1, 2, \dots, n]$$

is central. Indeed, it is easy to see that there is exactly one vertex-disjoint path system from  $[n]$  to  $[n]$  in  $G_\emptyset^{n \times n}$ , namely  $\mathcal{P} = (P_1, \dots, P_n)$ , where  $P_i = (i, (i, i), i)$  for each  $i \in [n]$ . Hence,

$$D_q = t_{1,1}t_{2,2} \cdots t_{n,n}.$$

Centrality of  $D_q$  follows from the observation that the right hand side commutes with every generator  $t_{i,j}^{\pm 1}$  of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ .

The next result was first proved in [3], but under the additional assumption that  $q$  is transcendental over  $\mathbb{Q}$ . We here provide a proof for the more general case that  $q$  is a nonzero, non-root of unity.

**Theorem 4.1.9.** *A quantum minor  $[I \mid J]$  with maximum coordinate at most  $(r, s)$  equals zero if and only if there does not exist a vertex-disjoint path system from  $I$  to  $J$ , i.e., if and only if  $\Gamma_B^{(t)}(I \mid J) = \emptyset$ .*

*Proof.* If  $\Gamma_B^{(t)}(I \mid J) = \emptyset$ , then Theorem 4.1.7 implies that  $[I \mid J] = 0$ .

Now suppose  $\Gamma_B^{(t)}(I \mid J) \neq \emptyset$ , i.e., there is at least one vertex-disjoint path system from  $I$  to  $J$ . The weight of a vertex-disjoint path system  $\mathcal{P}$  is equal to  $q^\alpha \mathbf{t}^{M_{\mathcal{P}}} \in \mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$  for some integer  $\alpha$ , where

$$(M_{\mathcal{P}})_{i,j} = \begin{cases} 1 & \text{if there is a path in } \mathcal{P} \text{ with a } \Gamma\text{-turn at } (i, j); \\ -1 & \text{if there is a path in } \mathcal{P} \text{ with a } \mathbb{J}\text{-turn at } (i, j); \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if for any distinct  $\mathcal{P}, \mathcal{Q} \in \Gamma_B^{(t)}(I \mid J)$  one has  $M_{\mathcal{P}} \neq M_{\mathcal{Q}}$ , then by Theorem 4.1.7 and Proposition 2.1.10, we may conclude that  $[I \mid J] \neq 0$ .

Suppose  $\mathcal{P} = (P_1, \dots, P_k)$  and  $\mathcal{Q} = (Q_1, \dots, Q_k)$  are two vertex-disjoint path systems from  $I$  to  $J$  and that  $M_{\mathcal{P}} = M_{\mathcal{Q}}$ , i.e., a path in  $\mathcal{P}$  has a  $\Gamma$ -turn (respectively  $\mathbb{J}$ -turn) at  $(i, j)$  if and only if a path in  $\mathcal{Q}$  does. We aim to show that  $\mathcal{P} = \mathcal{Q}$ . First, consider the paths  $P_k$  and  $Q_k$ . Let  $(i_k, \ell)$  be the first vertex where  $P_k$  turns, and  $(i_k, \ell')$  be the first vertex where  $Q_k$  turns. If  $\ell > \ell'$ , then  $Q_k$  goes straight through  $(i_k, \ell')$ . However, since  $\mathcal{Q}$  contains some path  $Q$  that turns at  $(i, \ell)$ , this implies (since  $B$  is a Cauchon diagram)

that  $Q$  and  $Q_k$  intersect, contradicting the choice of  $\mathcal{Q}$  as a vertex-disjoint path system. The symmetric case shows that  $\ell \not\prec \ell'$  and hence  $\ell = \ell'$ . A similar argument can then be applied to the remainder of the turning vertices (if any) in  $P_k$  and  $Q_k$ , from which we conclude that  $P_k = Q_k$ . Repeating the argument with  $P_{k-1}$  and  $Q_{k-1}$ , etc., we see that  $\mathcal{P} = \mathcal{Q}$ , as desired.  $\square$

**Corollary 4.1.10.** (Recall the map  $\sigma_B^{(t)} : A^{(t)} \rightarrow A_B^{(t)}$  of Section 3.3.) A quantum minor  $[I | J]^{(t)} \in A^{(t)}$  with maximum coordinate at most  $(r, s)$  is in  $\ker(\sigma_B^{(t)})$  if and only if there does not exist a vertex-disjoint path system from  $I$  to  $J$  in  $G_B^{m \times n}$ .  $\square$

We conclude this section by showing how one may construct new vertex-disjoint path systems from  $I$  to  $J$  from old. First, suppose  $i$  is a row vertex and  $j$  is a column vertex in  $G_B^{m \times n}$ , and consider two paths  $P: i \rightarrow j$  and  $Q: i \rightarrow j$ . Now,  $P \cap Q \neq \emptyset$  since, for example, both paths contain row vertex  $i$  and column vertex  $j$ . Let  $(i = v_0, \dots, v_k = j)$  be the subsequence of all vertices that  $P$  and  $Q$  share. For each  $a \in [k]$ , let  $P_a$  (respectively  $Q_a$ ) denote the subpath of  $P$  (respectively  $Q$ ) starting at  $v_{a-1}$  and ending at  $v_a$ . If  $P_a \neq Q_a$ , then the first edge of  $P_a$  is perpendicular to the first edge of  $Q_a$ . If the first edge of  $P_a$  is horizontal, let us say that  $P_a$  is *above*  $Q_a$ , otherwise  $P_a$  is *below*  $Q_a$ . Consider the paths

$$U_a = \begin{cases} P_a & \text{if } P_a = Q_a, \\ P_a & \text{if } P_a \text{ is above } Q_a, \\ Q_a & \text{if } Q_a \text{ is above } P_a, \end{cases}$$

and

$$L_a = \begin{cases} P_a & \text{if } P_a = Q_a, \\ P_a & \text{if } P_a \text{ is below } Q_a, \\ Q_a & \text{if } Q_a \text{ is below } P_a. \end{cases}$$

Now, let  $U(P, Q): i \rightarrow j$  be the path

$$U(P, Q) = U_1 \cup U_2 \cup \dots \cup U_k$$

and  $L(P, Q): i \rightarrow j$  be the path

$$L(P, Q) = L_1 \cup L_2 \cup \dots \cup L_k.$$

**Example 4.1.11.** With respect to Figure 10,  $U_1$  is the solid path from  $i = v_0$  to  $v_1$ ,  $U_2$  is the dashed path from  $v_1$  to  $v_2$ ,  $U_3$  is the solid path from  $v_2$  to  $v_3$ , etc. On the other hand,  $L_1$  is the solid path from  $i = v_0$  to  $v_1$ ,  $L_2$  is the solid path from  $v_1$  to  $v_2$ ,  $L_3$  is the solid path from  $v_2$  to  $v_3$ , etc.

The key property of  $U(P, Q)$  is the following.

**Lemma 4.1.12.** For a row vertex  $i$  and column vertex  $j$  in  $G_B^{m \times n}$ , consider two paths  $P: i \rightarrow j$  and  $Q: i \rightarrow j$ . Suppose that  $R: i' \rightarrow j'$  is a path with  $i' > i$ . If  $R$  is disjoint from either  $P$  or  $Q$ , then  $R$  is disjoint from  $U(P, Q)$ .

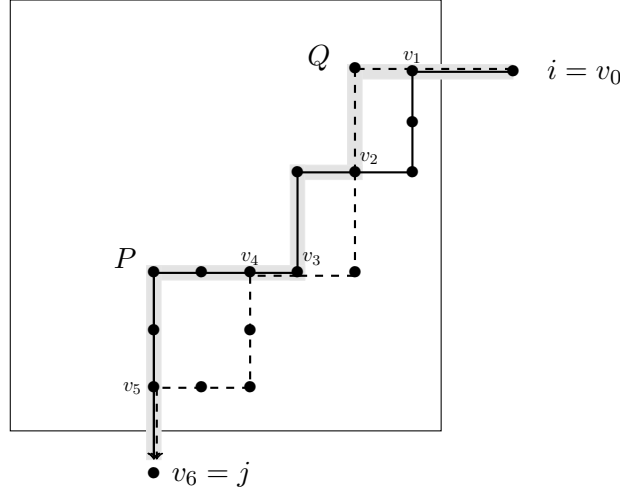


FIGURE 10.  $P$  is the solid path;  $Q$  is the dashed path;  $U(P, Q)$  is the shadowed path.

*Proof.* With respect to  $P$  and  $Q$ , we use the notation of the paragraph just prior to Example 4.1.11. Without loss of generality, suppose  $P$  and  $R$  are disjoint.

If  $R$  and  $U(P, Q)$  have a vertex  $w$  in common, then  $w \in Q$  and there exists an  $a$  such that  $w$  is in the subpath  $Q_a$  of  $Q$ . Since  $w \in U(P, Q)$ , we have  $U_a = Q_a$  for this  $a$  and so  $Q_a$  is above  $P_a$ . On the other hand, the plane curve formed by  $P_a$  and  $Q_a$  is closed, and since  $i > i'$ ,  $R$  must intersect this curve. Since  $G_B^{m \times n}$  is planar, the intersection occurs at a vertex of  $P$ , a contradiction to the assumed disjointness of  $P$  and  $R$ .  $\square$

**Corollary 4.1.13.** *Let  $i < i'$  be two row vertices and  $j < j'$  be two column vertices in  $G_B^{m \times n}$ . Suppose  $P: i \rightarrow j$  and  $P': i' \rightarrow j'$  are disjoint paths and  $Q: i \rightarrow j$  and  $Q': i' \rightarrow j'$  are disjoint paths. Then  $U(P, Q)$  and  $U(P', Q')$  are disjoint.*

*Proof.* By two applications of Lemma 4.1.12,  $U(P, Q)$  is disjoint from both  $P'$  and  $Q'$ . Since  $U(P', Q')$  consists only of subpaths coming from either  $P'$  or  $Q'$ , we have that  $U(P, Q)$  and  $U(P', Q')$  are disjoint as well.  $\square$

Repeated application of Corollary 4.1.13 immediately gives the following result.

**Corollary 4.1.14.** *Let  $\mathcal{P} = (P_1, \dots, P_k)$  and  $\mathcal{Q} = (Q_1, \dots, Q_k)$  be vertex-disjoint path systems from  $I$  to  $J$ . Then*

$$U(\mathcal{P}, \mathcal{Q}) = (U(P_1, Q_1), \dots, U(P_k, Q_k))$$

*is a vertex-disjoint path system from  $I$  to  $J$ .*  $\square$

Now, if  $\Gamma_B^{(t)}(I|J)$  is non-empty, then repeated applications of Corollary 4.1.14 to the finitely many path systems in  $\Gamma_B^{(t)}(I|J)$  shows that the next definition is sensible.

**Definition 4.1.15.** If  $\Gamma_B^{(t)}(I|J) \neq \emptyset$ , then the *supremum* of  $\Gamma_B^{(t)}(I|J)$  is the (unique) vertex disjoint path system  $(Q_1, \dots, Q_k) \in \Gamma_B^{(t)}(I|J)$  such that for any  $\mathcal{P} = (P_1, \dots, P_k) \in \Gamma_B^{(t)}(I|J)$  one has, for each  $i \in [k]$ ,

$$U(Q_i, P_i) = Q_i.$$

For  $L(P, Q)$ , it is clear that results similar to Lemma 4.1.12, Corollary 4.1.13 and Corollary 4.1.14 hold. We omit their explicit statements here, but note that the next definition is also sensible.

**Definition 4.1.16.** If  $\Gamma_B^{(t)}(I|J) \neq \emptyset$ , then the *infimum* of  $\Gamma_B^{(t)}(I|J)$  is the (unique) vertex disjoint path system  $(Q_1, \dots, Q_k) \in \Gamma_B^{(t)}(I|J)$  such that for any  $\mathcal{P} = (P_1, \dots, P_k) \in \Gamma_B^{(t)}(I|J)$  one has, for each  $i \in [k]$ ,

$$L(Q_i, P_i) = Q_i.$$

**Example 4.1.17.** Once again, consider the Cauchon graph of Figure 9. The supremum of  $\Gamma_B^{(16)}(1, 3 | 1, 3)$  is the path system  $(\tilde{Q}_1, \tilde{Q}_2)$  where

$$\begin{aligned}\tilde{Q}_1 &= (1, (1, 3), (1, 2), (2, 2), (4, 2), (4, 1), 1), \\ \tilde{Q}_2 &= (3, (3, 4), (3, 3), (4, 3), 3),\end{aligned}$$

while the infimum of  $\Gamma_B^{(16)}(1, 3 | 1, 3)$  is the path system  $(Q_1, Q_2)$ , where

$$\begin{aligned}Q_1 &= (1, (1, 3), (2, 3), (2, 2), (4, 2), (4, 1), 1), \\ Q_2 &= (3, (3, 4), (4, 4), (4, 3), 3).\end{aligned}$$

**4.2. Gröbner Bases.** In this section we review the theory of Gröbner basis theory as it applies to the algebras  $A^{(t)}$ . Gröbner basis theory is rather well-known in commutative algebra and fortunately many of the key aspects transfer to quantum matrices and the algebras  $R^{(t)} \simeq A^{(t)}$ . The main reference for this section is the book of Bueso, Gómez-Torrecillas and Verschoren [2].

Throughout this section, we fix  $t \in [mn]$ , let  $(r, s)$  be the  $t^{\text{th}}$  smallest coordinate, and denote the matrix of generators of  $A^{(t)}$  by  $[x_{i,j}]$ . We now define a total order of the lexicographic monomials in  $A^{(t)}$ .

**Definition 4.2.1.** The *matrix lexicographic order*  $\prec$  on  $\mathcal{M}_{m,n}(\mathbb{Z})$  is defined as follows. If  $M \neq N \in \mathcal{M}_{m,n}(\mathbb{Z})$ , let  $(k, \ell)$  be the least coordinate (with respect to the lexicographic order) in which  $M$  and  $N$  differ. Then we set

$$M \prec N \Leftrightarrow (M)_{k,\ell} < (N)_{k,\ell}$$

and say that  $M \prec N$  at  $(k, \ell)$ .

If  $M \prec N$  are both in  $\mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ , then the matrix lexicographic order induces a total order (also called matrix lexicographic) on the monomials of  $A^{(t)}$  by setting

$$\mathbf{x}^M \prec \mathbf{x}^N \Leftrightarrow M \prec N.$$

By allowing the  $(r, s)$ -entry in  $M$  and  $N$  to be negative, this terminology extends to a total order on the lexicographic monomials of  $A^{(t)}[x_{r,s}^{-1}]$ .

For example, under the matrix lexicographic order, we have

$$x_{i,j} \prec x_{k,\ell} \Leftrightarrow (i,j) > (k,\ell).$$

If  $(i,j), (k,\ell) \leq (r,s)$ , and  $(i,j)$  is northwest of  $(k,\ell)$ , then recall the relation

$$x_{k,\ell}x_{i,j} = x_{i,j}x_{k,\ell} - (q - q^{-1})x_{i,\ell}x_{k,j}.$$

On the other hand, we also have

$$x_{i,\ell}x_{k,j} \prec x_{i,j}x_{k,\ell}.$$

Essentially by repeated application of these facts and the other relations amongst the standard generators, we obtain the following, which is a special case of the more general Proposition 2.4 in [2].

**Proposition 4.2.2.** *For  $M, N \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ , the lexicographic expression of  $\mathbf{x}^M \mathbf{x}^N$  is*

$$\mathbf{x}^M \mathbf{x}^N = q^\alpha \mathbf{x}^{M+N} + \sum_{L \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})} \alpha_L \mathbf{x}^L,$$

for some integer  $\alpha$  and where for every  $\alpha_L \neq 0$ , one has  $L \prec M + N$ .  $\square$

**Definition 4.2.3.** Let  $M, N \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ . We say that  $\mathbf{x}^M$  divides  $\mathbf{x}^N$  if  $(M)_{i,j} \leq (N)_{i,j}$  for all  $(i,j) \in [m] \times [n]$ .

Using this terminology, we will use Proposition 4.2.2 in the following way.

**Corollary 4.2.4.** *Let  $M, N \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ . If  $\mathbf{x}^M$  divides  $\mathbf{x}^N$ , then there exists an integer  $\alpha$ , matrices  $L \prec N$ , and scalars  $\alpha_L \in \mathbb{K}^*$  such that*

$$\mathbf{x}^N = q^\alpha \mathbf{x}^M \mathbf{x}^{N-M} + \sum_L \alpha_L \mathbf{x}^L. \quad \square$$

**Remark 4.2.5.** Proposition 4.2.2, Definition 4.2.3 and Corollary 4.2.4 extend to  $A^{(t)}[x_{r,s}^{-1}]$  by allowing the  $(r,s)$ -entry in each matrix to be negative.

**Definition 4.2.6.** Let  $a \in A^{(t)}$  with lexicographic expression

$$a = \sum_L \alpha_L \mathbf{x}^L.$$

The *leading term* of  $a$  is the maximum lexicographic term with respect to the matrix lexicographic order that appears in the lexicographic expression. We denote the leading term of  $a$  by  $\ell t(a)$ .

We are now ready to give the definition of a Gröbner basis for a left ideal.

**Definition 4.2.7.** Let  $J$  be a left ideal of  $A^{(t)}$ , and let

$$G = \{g_1, g_2, \dots, g_k\} \subseteq J.$$

We say that  $G$  is a *Gröbner basis* for  $J$  if for every  $a \in J$  there exists a  $g_i \in G$  such that  $\ell t(g_i)$  divides  $\ell t(a)$ .

The key idea of a Gröbner basis is that one may find an expression for any  $a \in J$  as a combination of the  $g_i$  recursively. If  $\ell t(a)$  is divided by  $\ell t(g_i)$ , then by Corollary 4.2.4 we may write

$$a = g_i a' + b$$

where  $\ell t(b) \prec \ell t(a)$ . Since it follows that  $b \in J$ , we can repeat the process if  $b \neq 0$ . Since there are only finitely many lexicographic monomials smaller than  $\ell t(a)$ , this process will end after finitely many steps. Thus, the elements of the Gröbner basis are a generating set for  $J$ .

We will eventually apply the notions of this section to situations involving quantum minors, and we will need slightly more detailed information than is contained in Corollary 4.2.4. We provide this information in Part 2 of the next result.

**Lemma 4.2.8.** Let  $[I | J]^{(t)} \in A^{(t)}$  be a minor with maximum coordinate  $(i_k, j_k)$ . Recalling Remark 4.1.2, if we write

$$[I | J]^{(t)} = \sum_{\sigma \in S_k} (-q)^{\ell(\sigma)} \mathbf{x}^{P_\sigma},$$

then:

- (1) One has  $\ell t([I | J]^{(t)}) = \mathbf{x}^{P_{\text{id}}}$ , where  $\text{id}$  is the identity permutation;
- (2) If  $\mathbf{x}^{P_{\text{id}}}$  divides  $\mathbf{x}^L$  for some  $L \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$ , then

$$\mathbf{x}^L = q^\alpha [I | J]^{(t)} \mathbf{x}^{L - P_{\text{id}}} + w, \tag{5}$$

for some integer  $\alpha$  and element  $w \in A^{(t)}$  where, if  $\ell t(w) = \mathbf{x}^K$ , then  $K \prec L$  at a coordinate northwest of  $(i_k, j_k)$ .

*Proof.* Part 1 is clear as  $P_\sigma \prec P_{\text{id}}$  at the least diagonal coordinate  $(i_{k'}, j_{k'})$  of  $[I | J]^{(t)}$  with  $\sigma(k') \neq k'$ .

For Part 2, note that the generators with coordinates west, north or northwest of  $(i_k, j_k)$  commute or  $q^{\pm 1}$ -commute with the generators northeast of  $(i_k, j_k)$ . Therefore, we may write

$$\mathbf{x}^L = q^\alpha \mathbf{x}^{P_{\text{id}}} \mathbf{x}^{L - P_{\text{id}}} + \sum_{N \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})} \alpha_N \mathbf{x}^N,$$

for some integer  $\alpha$ , where, if  $\alpha_N \neq 0$ , then the entries of  $N$  and  $L$  are equal in all coordinates that are *not* north, west, or northwest of  $(i_k, j_k)$ . On the other hand, since all commutation relations for  $A^{(t)}$  are homogeneous, it follows that  $L$  and  $N$  have the same bidegree (i.e., respective row and column sums), so that the first entry in which  $L$  and  $N$  differ must actually be northwest of  $(i_k, j_k)$ .



Next, we have

$$\mathbf{x}^{P_{\text{id}}} \mathbf{x}^{L-P_{\text{id}}} = [I | J]^{(t)} \mathbf{x}^{L-P_{\text{id}}} - \sum_{\sigma \in S_k \setminus \text{id}} (-q)^{\ell(\sigma)} \mathbf{x}^{P_\sigma} \mathbf{x}^{L-P_{\text{id}}}.$$

If  $\sigma \neq \text{id}$ , then

$$\mathbf{x}^{P_\sigma} \mathbf{x}^{L-P_{\text{id}}} = \mathbf{x}^{L-P_{\text{id}}+P_\sigma} + w',$$

where, by reasoning similar to that in the previous paragraph, every  $\mathbf{x}^K$  appearing in  $w'$  satisfies  $K \prec L - P_{\text{id}} + P_\sigma$  at a coordinate northwest of  $(i_k, j_k)$ . Finally,  $L$  and  $L - P_{\text{id}} + P_\sigma$  also differ at an entry northwest of  $(i_k, j_k)$ . Combining the facts of this paragraph and the preceding one, we obtain Equation (5) with the desired property of  $w$ .  $\square$

**4.3. Adding Derivations and Lexicographic Expressions.** Throughout this section, we fix  $t \in [mn], t \neq 1$  and let  $(r, s)$  be the  $t^{\text{th}}$  smallest coordinate. Let  $[x_{i,j}]$  be the matrix of generators for  $A^{(t)}$ , and  $[y_{i,j}]$  the matrix of generators for  $A^{(t-1)}$ .

The proof of Theorem 4.4.1 requires a somewhat detailed understanding of the effect of the adding derivations map on the lexicographic expressions of an element  $a \in A^{(t)}$  and its image  $\overleftarrow{a} \in A^{(t-1)}[y_{r,s}^{-1}]$ . This short section provides this information.

Recall that the adding derivations map is the homomorphism

$$\overleftarrow{\cdot} : A^{(t)} \rightarrow A^{(t-1)}[y_{r,s}^{-1}]$$

defined on the standard generators by

$$\overleftarrow{x}_{i,j} = \begin{cases} y_{i,j} + y_{i,s} y_{r,s}^{-1} y_{r,j}, & \text{if } (i,j) \text{ is northwest of } (r,s); \\ y_{i,j}, & \text{otherwise,} \end{cases}$$

or, equivalently, by

$$\overleftarrow{x}_{i,j} = \begin{cases} y_{i,j} + q y_{i,s} y_{r,j} y_{r,s}^{-1}, & \text{if } (i,j) \text{ is northwest of } (r,s); \\ y_{i,j}, & \text{otherwise.} \end{cases}$$

Let  $\mathbf{x}^M \in A^{(t)}$  and write

$$\mathbf{x}^M = x_{i_1, j_1} x_{i_2, j_2} \cdots x_{i_p, j_p},$$

where for each  $k \in [p-1]$ ,  $(i_k, j_k) \leq (i_{k+1}, j_{k+1})$ . Thus,

$$\overleftarrow{\mathbf{x}}^M = \sum_C q^{|C|} \overleftarrow{x}_{i_1, j_1}^C \overleftarrow{x}_{i_2, j_2}^C \cdots \overleftarrow{x}_{i_p, j_p}^C, \quad (6)$$

where the sum is over all subsets  $C$  of the set of  $k$  where  $(i_k, j_k)$  is northwest of  $(r, s)$ , and where, for such a  $C$ ,

$$\overleftarrow{x}_{i_k, j_k}^C = \begin{cases} y_{i_k, s} y_{r, j_k} y_{r, s}^{-1}, & \text{if } k \in C; \\ y_{i_k, j_k}, & \text{if } k \notin C. \end{cases}$$

**Lemma 4.3.1.** *With notation as in the preceding discussion, let  $z$  be a monomial on the right side of Equation (6), so that for some  $C$ ,*

$$z = \overleftarrow{x_{i_1, j_1}}^C \overleftarrow{x_{i_2, j_2}}^C \cdots \overleftarrow{x_{i_p, j_p}}^C.$$

*Consider the lexicographic expression of  $z$ , written as*

$$z = \sum_{L_C \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})} \alpha_{L_C} \mathbf{y}^{L_C}$$

*where  $\alpha_{L_C} \in \mathbb{K}^*$ . Then the following hold:*

(1) *For each  $L_C$ ,*

$$(L_C)_{r,s} = (M)_{r,s} - |C|.$$

(2) *If  $C \neq \emptyset$ , then for every  $L_C$ , we have  $L_C \prec M$  at the least  $(i_k, j_k)$  for which  $k \in C$ .*

(3) *For each  $L_C$  and for each  $i \in [m] \setminus r$ ,*

$$(L_C)_{i,s} = (M)_{i,s} + |\{k \in C \mid i_k = i\}|.$$

(4) *If  $(i, j)$  is northwest of  $(r, s)$  and if*

$$(L_C)_{i,j} > (M)_{i,j} - |\{k \in C \mid (i_k, j_k) = (i, j)\}|,$$

*then there is a coordinate  $(i, j')$  with  $1 \leq j' < j$  such that*

$$(L_C)_{i,j'} < (M)_{i,j'} - |\{k \in C \mid (i_k, j_k) = (i, j')\}|.$$

(5) *For each  $L_C$ , the entries in coordinates not north, west or northwest of  $(r, s)$  are equal to the corresponding entries in  $M$ .*

*Proof.* First, let us split the monomial  $z$  by row indices, i.e., write

$$z = (\overleftarrow{x_{1,j_{1,1}}}^C \overleftarrow{x_{1,j_{1,2}}}^C \cdots \overleftarrow{x_{1,j_{1,p_1}}}^C) \cdots (\overleftarrow{x_{m,j_{m,1}}}^C \overleftarrow{x_{m,j_{m,2}}}^C \cdots \overleftarrow{x_{m,j_{m,p_m}}}^C),$$

where, for each  $i \in [m]$ , the generators appearing in the monomial

$$\overleftarrow{x_{i,j_{i,1}}}^C \overleftarrow{x_{i,j_{i,2}}}^C \cdots \overleftarrow{x_{i,j_{i,p_i}}}^C$$

have indices

$$(a, b) \in \{(i, j) \mid j \in [n]\} \cup \{(r, j) \mid j \in [s]\}.$$

Moreover, if  $y_{r,j}$  appears with  $j \neq s$ , then  $y_{r,j}$  is to the right of any  $y_{i,j'}$  with  $j' < j$ . In other words, such a  $y_{r,j}$  commutes or  $q$ -commutes with every generator appearing to its right. Since  $y_{r,s}^{-1}$  commutes or  $q^{\pm 1}$ -commutes with every generator of  $A^{(t-1)}$ , we may write

$$\overleftarrow{x_{i,j_{i,1}}}^C \overleftarrow{x_{i,j_{i,2}}}^C \cdots \overleftarrow{x_{i,j_{i,p_i}}}^C = q^\alpha \mathbf{y}^{M_i} \mathbf{y}^{R_i} y_{r,s}^{-\beta},$$

where  $\alpha \in \mathbb{Z}$ ,  $\beta$  is the number of occurrences of  $y_{r,s}^{-1}$  in the left monomial,  $M_i \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$  is the matrix defined by

$$(M_i)_{a,b} = \begin{cases} 0 & \text{if } a \neq i; \\ (M)_{i,b} - |\{k \in C \mid (i_k, j_k) = (i, b)\}| & \text{if } a = i \text{ and } 1 \leq b < s; \\ (M)_{i,s} + |\{k \in C \mid i_k = i\}| & \text{if } a = i \text{ and } b = s; \\ (M)_{i,b} & \text{if } s < b \leq n, \end{cases}$$

and  $R_i$  is a matrix whose nonzero entries appear only in coordinates between  $(r, 1)$  and  $(r, s - 1)$ .

It follows that we may write

$$z = q^{\alpha'} \mathbf{y}^{M_1} \mathbf{y}^{R_1} \mathbf{y}^{M_2} \mathbf{y}^{R_2} \dots \mathbf{y}^{M_{r-1}} \mathbf{y}^{R_{r-1}} \mathbf{y}^{R_r} y_{r,s}^{-|C|} \mathbf{y}^L, \quad (7)$$

for some  $\alpha' \in \mathbb{Z}$ , where the entries of  $R_r$  equal those of  $M$  at coordinates between  $(r, 1)$  and  $(r, s - 1)$  and are zero elsewhere, and where entries of  $L$  equal those of  $M$  at all coordinates greater than  $(r, s)$ .

Next, let  $y_{r,j}$  be a generator with  $1 \leq j < s$ , and consider  $y_{r,j} \mathbf{y}^{M_i}$  for some  $1 \leq i < r$ . Recall that, for  $j' < j$ , we have the relation

$$y_{r,j} y_{i,j'} = y_{i,j'} y_{r,j} - (q - q^{-1}) y_{i,j} y_{r,j'}.$$

Repeated applications of this relation imply that

$$y_{r,j} \mathbf{y}^{M_i} = \mathbf{y}^{M_i} y_{r,j} + \sum_{\ell} \alpha_{\ell} \mathbf{y}^{M_i^{\ell}} \mathbf{y}^{R^{\ell}},$$

for nonzero scalars  $\alpha_{\ell}$  and where:

- (1) Every  $M_i^{\ell} \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$  satisfies  $M_i^{\ell} \prec M_i$ , and the entries of each  $M_i^{\ell}$  differ from those in  $M_i$  only between coordinates  $(i, 1)$  and  $(i, s - 1)$ ;
- (2) Each  $R^{\ell} \in \mathcal{M}_{m,n}(\mathbb{Z}_{\geq 0})$  has nonzero entries only between coordinates  $(r, 1)$  and  $(r, s - 1)$ .

In particular, when finding the lexicographic expression of the monomial  $z$  written in the form of Equation (7), we never create or destroy any of the generators  $y_{i,s}$ ,  $y_{r,s}^{\pm 1}$ , nor any generator with coordinates *not* north, west or northwest. Parts 1, 3 and 5 of the lemma follow. It also follows that for every  $L_C$  and  $i \in [r - 1]$ , if the entries in  $L_C$  and  $M$  with coordinates between  $(i, 1)$  and  $(i, s - 1)$  differ, then the first different entry is smaller in  $L_C$ . This implies Part 4. Finally, Part 2 comes from the fact that each term in the lexicographic expression of  $z$  must start with  $y_{i_1, j_1} \dots y_{i_{k-1}, j_{k-1}}$  since no subsequent relation produces a generator  $y_{a,b}$  with  $(a, b) < (i_k, j_k)$ .  $\square$

**Corollary 4.3.2.** *If  $a \in A^{(t)}$ , and  $\ell t(a) = \mathbf{x}^M$ , then  $\ell t(\overleftarrow{a}) = \mathbf{y}^M$ .*

*Proof.* If  $C \neq \emptyset$ , then each term  $\mathbf{y}^{L_C}$  in the resulting lexicographic expression satisfies  $\mathbf{y}^{L_C} \prec \mathbf{y}^M$  by Part 2 of Lemma 4.3.1. On the other hand,

$$\mathbf{y}^M = \overleftarrow{x_{i_1, j_1}} \overleftarrow{x_{i_2, j_2}} \dots \overleftarrow{x_{i_p, j_p}}.$$

□

**4.4. Generators of  $\mathcal{H}$ -primes.** Throughout this section, we fix a Cauchon diagram  $B$  and, recalling Section 3.3, consider the sequence of  $\mathcal{H}$ -primes  $(K_1, \dots, K_{mn})$ , where

$$K_t = \ker \left( \sigma_B^{(t)} \right).$$

**Theorem 4.4.1.** *For  $t \in [mn]$ , let  $(r, s)$  be the  $t^{\text{th}}$  smallest coordinate and let  $[x_{i,j}]$  be the matrix of generators for  $A^{(t)}$ . If  $G_t$  is the subset of  $K_t$  consisting of all  $x_{i,j}$  with  $(i, j) > (r, s)$  and  $(i, j) \in B$ , together with all quantum minors in  $K_t$  whose maximum coordinate is at most  $(r, s)$ , then  $G_t$  is a Gröbner basis for  $K_t$ .*

*Proof.* Note that we will repeatedly use Theorem 4.1.7 and Corollary 4.1.10 without explicit reference to them. We proceed by induction on  $t$ .

If  $t = 1$ , then the only minor in  $A^{(1)} = \mathcal{O}_q(\mathbb{K}^{m \times n})$  whose maximum coordinate is  $(1, 1)$  is

$$[1 \mid 1]^{(1)} = t_{1,1}.$$

On the other hand,  $t_{1,1} \in K_1$  if and only if  $(1, 1) \in B$ . So  $G_1$  is precisely the set of generators  $t_{i,j}$  with  $(i, j) \in B$ . On the other hand, these  $t_{i,j}$  generate  $K_1$  by Theorem 2.3.4 and so Proposition 2.1.10 implies  $G_1$  is indeed a Gröbner basis.

So now suppose  $t \neq 1$  and that  $G_{t-1}$  is a Gröbner basis for  $K_{t-1}$ . Let  $[y_{i,j}]$  be the matrix of generators for  $A^{(t-1)}$ . There are two cases to consider, according to whether or not  $(r, s) \in B$ .

If  $(r, s) \in B$ , then, as elements of  $\mathcal{O}_q((\mathbb{K}^\times)^{m \times n})$ , we have for each coordinate  $(i, j)$  that

$$\sigma_B^{(t)}(x_{i,j}) = \sigma_B^{(t-1)}(y_{i,j}).$$

Therefore,

$$a = \sum_L \alpha_L \mathbf{x}^L \in K_t$$

if and only if

$$a' = \sum_L \alpha_L \mathbf{y}^L \in K_{t-1}.$$

Hence, if  $\mathbf{y}^M$  divides  $\ell t(a')$ , then  $\mathbf{x}^M$  divides  $\ell t(a)$ .

Now, the previous paragraph also implies that if  $[I \mid J]^{(t-1)} \in K_{t-1}$  with maximum coordinate at most  $(r, s)^-$ , then  $[I \mid J]^{(t)} \in K_t$  with maximum coordinate strictly less than  $(r, s)$  so that  $[I \mid J]^{(t)} \in G_t$ . Also, if  $(i, j) > (r, s)$  is such that  $(i, j) \in B$ , then  $x_{i,j} \in K_t$ . Finally, since  $(r, s) \in B$ ,

$$[r \mid s]^{(t)} = x_{r,s} \in K_t.$$

It now follows that since  $G_{t-1}$  is a Gröbner basis for  $K_{t-1}$ ,  $G_t$  is a Gröbner basis<sup>3</sup> for  $K_t$ .

---

<sup>3</sup>In general we have actually shown that a subset of  $G_t$  is a Gröbner basis for  $K_t$ , but nothing is lost by adding the extra minors in  $K_t$  with maximum coordinate equal to  $(r, s)$ .

Now let us assume  $(r, s) \notin B$ , i.e.,  $x_{r,s} \notin K_t$ , and fix a monic  $a \in K_t$  with lexicographic expression

$$a = \mathbf{x}^M + \sum_L \alpha_L \mathbf{x}^L,$$

where  $\ell t(a) = \mathbf{x}^M$ , i.e.,  $L \prec M$  for every  $L$  with a nonzero coefficient. Furthermore, we may assume that  $a$  is homogeneous with respect to the bigrading introduced at the end of Section 2.1, i.e., we may assume that, for each  $i \in [m]$ , the  $i^{\text{th}}$  row sum of every  $L$  and  $M$  are equal, and for every  $j \in [n]$ , the  $j^{\text{th}}$  column sum of  $M$  and every  $L$  are equal.

If there exists an  $(i, j) \in B$  with  $(i, j) > (r, s)$  and  $(M)_{i,j} \geq 1$ , then  $x_{i,j} \in G_t$  divides  $\ell t(a)$ , and we are done. So we may assume no such  $(i, j)$  exists. In fact by Lemma 3.3.4 we may further assume that  $M$  and every  $L$  have the same values in each coordinate  $(i, j) > (r, s)$ , and, without loss of generality, that these entries are all zero, i.e.,  $(M)_{i,j} = 0 = (L)_{i,j}$  for all  $(i, j) > (r, s)$ .

Since  $(r, s) \notin B$ ,

$$K_t = \overrightarrow{K_{t-1}}[x_{r,s}^{-1}] \cap A^{(t)},$$

and so there exists a  $b \in K_{t-1}$  and a nonnegative integer  $h$  with

$$a = \overrightarrow{b} x_{r,s}^{-h} \Leftrightarrow b = \overleftarrow{a} y_{r,s}^h.$$

By Corollary 4.3.2,

$$\ell t(b) = \mathbf{y}^M y_{r,s}^h.$$

We henceforth call a minor in  $G_{t-1}$  whose leading term divides  $\ell t(b)$  *critical*. Note that since the maximum coordinate of a critical minor is at most  $(r, s)^-$ , its leading term actually divides  $\mathbf{y}^M$ . By induction, there exists at least one critical minor. Now, if  $[I | J]^{(t-1)}$  is critical and  $[I | J]^{(t)} \in K_t$ , then, since the maximum coordinate of  $[I | J]^{(t)}$  is strictly less than  $(r, s)$ , we have found an element of  $G_t$  whose leading term divides  $\ell t(a)$ , and there is nothing left to prove. *From now on, we assume that if  $[I | J]^{(t-1)}$  is critical, then  $[I | J]^{(t)} \notin K_t$ .*

**Claim 1.** *If  $[I | J]^{(t-1)}$  is critical, where  $I = (i_1 < i_2 < \dots < i_k)$  and  $J = (j_1 < j_2 < \dots < j_k)$ , then we may assume the following.*

- (1) *Every vertex-disjoint path system in  $\Gamma_B^{(t)}(I | J) \neq \emptyset$  contains a path with a  $\mathbf{I}$ -turn at  $(r, s)$ .*
- (2) *If  $(i_{k'}, j_{k'})$  is the largest diagonal coordinate northwest of  $(r, s)$ , then*

$$[i_1, \dots, i_{k'} | j_1, \dots, j_{k'}]^{(t-1)}$$

*is critical.*

- (3) *If  $(i_k, j_k)$  is northwest of  $(r, s)$ , then for every  $(i, j)$  with  $i_k < i \leq r$  and  $j_k < j \leq s$ , one has  $(M)_{i,j} = 0$ .*

*Proof of Claim 1:*

*Part 1:* This is simply restating the assumption preceding the claim, since otherwise there is a vertex-disjoint path system in  $\Gamma_B^{(t-1)}(I|J)$ , i.e.,

$$[I|J]^{(t-1)} \notin K_{t-1}.$$

*Part 2:* By Part 1, there exists a  $\mathbb{J}$ -turn at  $(r, s)$  in any vertex-disjoint path system in  $\Gamma_B^{(t)}(I|J)$ . Hence  $s \notin J$  and  $(i_1, j_1)$  is northwest of  $(r, s)$ .

If  $(i_k, j_k)$  is northwest of  $(r, s)$ , then there is nothing to prove, so suppose  $(i_k, j_k)$  is northeast of  $(r, s)$ . If  $[I \setminus i_k | J \setminus j_k]^{(t-1)} \in K_{t-1}$ , then replace  $[I|J]^{(t-1)}$  with  $[I \setminus i_k | J \setminus j_k]^{(t-1)}$  and restart this argument. So assume that  $(i_k, j_k)$  is northeast of  $(r, s)$  and  $[I \setminus i_k | J \setminus j_k]^{(t-1)} \notin K_{t-1}$ , i.e., there exists a vertex-disjoint path system

$$\mathcal{P} = (P_1, \dots, P_{k-1}) \in \Gamma_B^{(t-1)}(I \setminus i_k | J \setminus j_k).$$

Let

$$\mathcal{Q} = (Q_1, \dots, Q_k) \in \Gamma_B^{(t)}(I|J).$$

From Part 1, there exists a  $Q_\alpha : i_\alpha \rightarrow j_\alpha$  containing  $(r, s)$  as a  $\mathbb{J}$ -turn. Clearly, we must have  $\alpha = k'$ , and since  $(i_k, j_k)$  is northeast of  $(r, s)$ ,  $k' \neq k$ . Recalling Corollary 4.1.14, consider the vertex-disjoint path system

$$\mathcal{R} = U(\mathcal{P}, \mathcal{Q} \setminus Q_k) \in \Gamma_B^{(t-1)}(I \setminus i_k | J \setminus j_k)$$

See Figure 11. Since  $P_{k'}$  does not contain a  $\mathbb{J}$ -turn at  $(r, s)$ , the path  $U(P_{k'}, Q_{k'})$  does not contain a  $\mathbb{J}$ -turn at  $(r, s)$ . Moreover, by Corollary 4.1.13,  $\mathcal{R}$  is disjoint from  $Q_k$ . Hence,  $\mathcal{R} \cup Q_k$  is a vertex-disjoint path system in the empty set  $\Gamma_B^{(t-1)}(I|J)$ , an impossibility.

*Part 3:* If  $(i, j) = (r, s)$  and  $(M)_{r,s} \geq 1$ , then  $[I \cup r | J \cup s]^{(t)}$  is a minor whose leading term divides  $\mathbf{x}^M$  with maximum coordinate  $(r, s)$ . The only path in  $\Gamma_B^{(t)}(r, s)$  is  $(r, (r, s), s)$ . Hence, if  $\Gamma_B^{(t)}(I \cup r | J \cup s)$  is nonempty, then any path system in this set would have a sub-path system from  $I$  to  $J$  not using  $(r, s)$ . But this is a vertex-disjoint path system in the empty set  $\Gamma_B^{(t-1)}(I|J)$ , an impossibility. Thus,  $[I \cup r | J \cup s]^{(t)} \in G_t$  with leading term dividing  $\mathbf{x}^M = \ell t(a)$ , and there is nothing left to prove. So we may assume  $(M)_{r,s} = 0$ .

If  $(i, j) \neq (r, s)$  but  $(M)_{i,j} \geq 1$ , then the leading term of  $[I \cup i | J \cup j]^{(t-1)}$  divides  $\mathbf{y}^M$ . Since  $[I|J]^{(t-1)} \in K_{t-1}$ , there is no vertex disjoint path system in  $\Gamma_B^{(t-1)}(I|J)$  and so certainly no vertex-disjoint path system in  $\Gamma_B^{(t-1)}(I \cup i | J \cup j)$ . Thus,  $[I \cup i | J \cup j]^{(t-1)}$  is critical and so there exists a  $\mathcal{P} \in \Gamma_B^{(t)}(I \cup i | J \cup j)$ . By Part 1 and vertex-disjointness, the path  $P : i \rightarrow j \in \mathcal{P}$  is necessarily the path with a  $\mathbb{J}$ -turn at  $(r, s)$ . But then  $\mathcal{P} \setminus \{P\}$  is a vertex-disjoint path system in the empty set  $\Gamma_B^{(t-1)}(I|J)$ , an impossibility. This completes the proof of the Claim.

We now say that a coordinate  $(i, j)$  is *critical* if  $(i, j)$  is northwest of  $(r, s)$  and there exists a critical minor with  $(i, j)$  as its maximum coordinate.

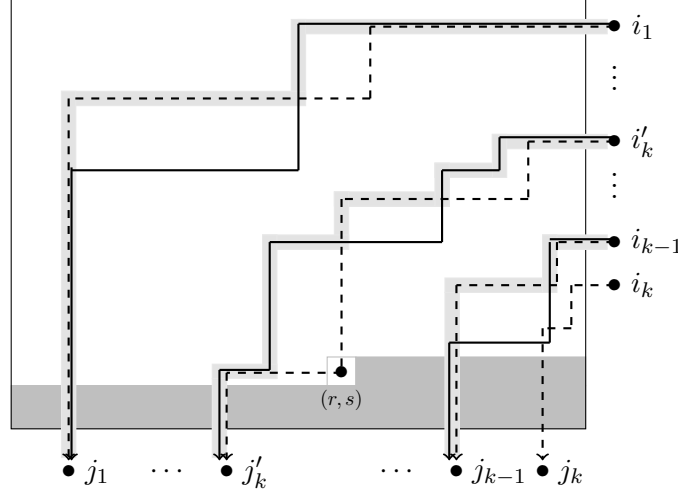


FIGURE 11. Illustration of the idea used to prove Part 2 of Claim 1. The dashed paths represent  $\mathcal{Q} \in \Gamma_B^{(t)}(I | J)$  containing a path with a J-turn at  $(r, s)$ . The solid paths represent a vertex-disjoint path system  $\mathcal{P} \in \Gamma_B^{(t-1)}(I \setminus i_k | J \setminus j_k)$ . The shaded paths represent  $U(\mathcal{P}, \mathcal{Q} \setminus Q_k)$ .

**Claim 2.** *If  $(i, j)$  is critical, then every  $(i, j')$  for  $j < j' < s$  with  $(M)_{i, j'} \geq 1$  is critical, and every  $(i', j)$  for  $i < i' < r$  with  $(M)_{i', j} \geq 1$  is critical.*

*Proof of Claim 2:* Suppose  $[I | J]^{(t-1)}$  is a critical minor whose maximum coordinate is  $(i, j)$ . Notice that the leading term of

$$[I | J \setminus j \cup j']^{(t-1)}$$

divides  $\mathbf{y}^M$  and its maximum coordinate is  $(i, j')$ , so it remains to show that this minor is in  $K_{t-1}$ .

Since  $[I | J]^{(t-1)}$  is critical, we may consider the supremum  $\mathcal{P} \in \Gamma_B^{(t)}(I | J) \neq \emptyset$ , which, by Part 1 of Claim 1, contains a path  $P : i \rightarrow j$  with a J-turn at  $(r, s)$ . Notice that  $P$  must have a horizontal subpath from  $(r, s)$  to  $(r, j)$ , followed by a  $\Gamma$ -turn at  $(r, j)$ , and then vertically down to the column vertex  $j$ . In particular,  $(r, j)$  is a white vertex. See Figure 12.

Suppose that  $[I | J \setminus j \cup j']^{(t-1)} \notin K_{t-1}$ , i.e., there exists a vertex-disjoint path system  $\mathcal{Q}$  from  $I$  to  $J \setminus j \cup j'$  in  $\Gamma_B^{(t-1)}(I | J \setminus j \cup j')$ . Therefore, the path  $Q : i \rightarrow j'$  in  $\mathcal{Q}$  does *not* use vertex  $(r, s)$ . By considering the appropriate supremums, we may assume without loss of generality that  $\mathcal{Q} \setminus Q = \mathcal{P} \setminus P$ . Now, since  $j' > j$ ,  $Q$  must intersect  $P$  in order to end at  $j'$ . Since  $Q$  cannot have a J-turn at a  $(r, s)$  or any larger vertex, the Cauchon condition implies that  $(r, j')$  is a white vertex. On the other hand,  $\mathcal{P} \setminus P$  is disjoint from both  $Q$  and  $P$ . If we let  $R$  be the path starting at  $i$ , equal to  $Q$  up to  $(r, j')$ , then

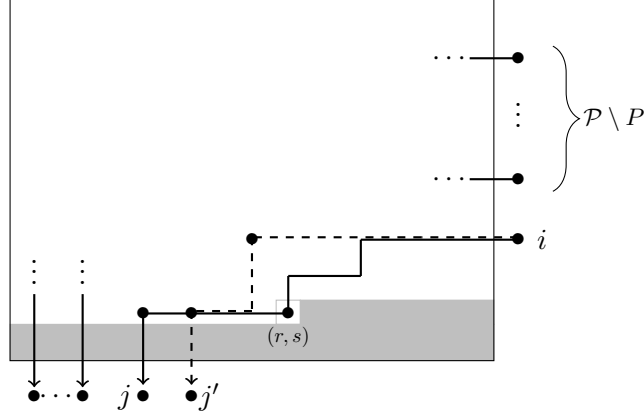


FIGURE 12. Illustration of the idea used in proving Claim 2. In the notation of that proof, the dashed line represents  $Q$  and the solid line represents  $P$ . The other vertices and partial paths represent  $\mathcal{P} \setminus P = \mathcal{Q} \setminus Q$ .

equal to  $P$  until the column vertex  $j$ , then  $R$  is a path from  $i$  to  $j$ , that does not contain  $(r, s)$ . Now  $(\mathcal{P} \setminus P) \cup R$  is a vertex-disjoint path system in  $\Gamma_B^{(t-1)}(I | J)$ , a contradiction. That a coordinate  $(i', j)$  with  $i < i' < r$  with  $(M)_{i', j} \geq 1$  is critical is proven similarly. This completes the proof of Claim 2.

To summarize the discussion so far, we have shown that it suffices to assume the following.

- If  $[I | J]^{(t-1)}$  is a critical minor, then  $\Gamma_B^{(t)}(I | J) \neq \emptyset$  and every vertex-disjoint path system contains a path with a J-turn at  $(r, s)$  (by Part 1 of Claim 1).
- Every critical minor contains a unique critical coordinate (by Part 2 of Claim 1).
- For each critical coordinate  $(i, j)$ , there is a critical minor whose maximum coordinate is  $(i, j)$  (by definition).
- For each critical coordinate  $(i, j)$  (of which there exists at least one),  $(M)_{k, \ell} = 0$  for all  $i < k \leq r$  and  $j < \ell \leq s$  (by Part 3 of Claim 1). In particular, no critical coordinate is northwest of another critical coordinate. See Figure 13.
- If  $(i, j)$  is *not* a critical coordinate, then no coordinate above or to its left is critical (by Claim 2).

The remainder of this proof will show that the above list of assumptions leads to a contradiction to the induction hypothesis.



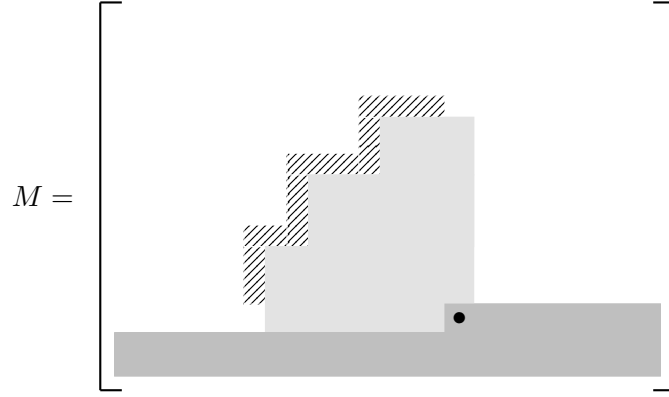


FIGURE 13. Structure of  $M$ : The bullet represents coordinate  $(r, s)$ . All critical coordinates lie in the striped region. All entries in the two regions shaded solid gray are 0.

Recalling the notation in Section 4.3, let

$$\ell t(a) = \mathbf{x}^M = x_{i_1, j_1} x_{i_2, j_2} \cdots x_{i_p, j_p},$$

and set

$$C = \{k \in [p] \mid (i_k, j_k) \text{ is critical}\} \neq \emptyset.$$

Consider the monomial

$$\overbrace{x_{i_1, j_1}}^C \overbrace{x_{i_2, j_2}}^C \cdots \overbrace{x_{i_p, j_p}}^C y_{r, s}^h.$$

By Lemma 4.3.1 and Proposition 4.2.2, the lexicographic expression of this monomial equals

$$q^\alpha \mathbf{y}^{N_C} + \sum_{L_C \in \mathcal{M}_{m, n}(\mathbb{Z})} \alpha_{L_C} \mathbf{y}^{L_C},$$

for some integer  $\alpha$  and with every  $L_C \prec N_C$  where

$$(N_C)_{i, j} = \begin{cases} 0, & \text{if } (i, j) \text{ is critical;} \\ (M)_{i, j}, & \text{if } i \neq r, j \neq s \text{ and } (i, j) \text{ not critical;} \\ (M)_{i, s} + \sum_{j'} (M)_{i, j'}, & \text{if } i \neq r \text{ and } j = s; \\ (M)_{r, j} + \sum_{i'} (M)_{i', j}, & \text{if } i = r \text{ and } j \neq s; \\ h - |C|, & \text{if } i = r \text{ and } j = s, \end{cases}$$

and where the sum in the case that  $i \neq r$  and  $j = s$  is over all  $j'$  with  $(i, j')$  critical, and the sum in the case that  $i = r$  and  $j \neq s$  is over all  $i'$  with  $(i', j)$  critical. With respect to Figure 13, the entries in the striped region are 0 in  $N_C$ , while entries above  $(r, s)$  (respectively to the left of  $(r, s)$ ) may become nonzero if there is a critical coordinate to the left (respectively above).

**Claim 3.** *The term  $\mathbf{y}^{N_C}$  is not divided by the leading term of any element of  $G_{t-1}$ . Consequently,  $\mathbf{y}^{N_C}$  is not the leading term of any element of  $K_{t-1}$ .*

*Proof of Claim 3:* To the contrary, suppose that the leading term of some element in  $G_{t-1}$  divides  $\mathbf{y}^{N_C}$ . Since  $(N_C)_{i,j} = (M)_{i,j} = 0$  for every  $(i,j) \geq (r,s)$ , this element is a minor

$$[I | J]^{(t-1)},$$

where, say,

$$I = (i_1 < \dots < i_z) \quad \text{and} \quad J = (j_1 < \dots < j_z).$$

If the leading term of  $[I | J]^{(t-1)}$  divides  $\mathbf{y}^M$ , then this minor contains a critical coordinate  $(i,j)$ . On the other hand,  $(N_C)_{i,j} = 0$  for any critical coordinate  $(i,j)$ , a contradiction. Therefore,  $[I | J]^{(t-1)}$  must have a coordinate  $(i_k, j_k)$  in which  $(N_C)_{i_k, j_k} > 0$  but  $(M)_{i_k, j_k} = 0$ . From the structure of the entries of  $N_C$ , there are only two possibilities: either  $(i_k, j_k) = (i_k, s)$  where  $(i_k, j'_k)$  is critical for some  $j'_k$ , or  $(i_k, j_k) = (r, j_k)$  where  $(i'_k, j_k)$  is critical for some  $i'_k$ . We here show that the former possibility leads to a contradiction. The latter case is dealt with similarly.

Before we begin, we simplify our presentation slightly by further assuming that  $(i_k, j_k) = (i_k, s)$  is the maximum coordinate of  $[I | J]^{(t-1)}$ , i.e., that  $z = k$ . The general case is obtained by simply adding in  $i_{k+1}, \dots, i_z$  and  $j_{k+1}, \dots, j_z$  to the respective index sets of every minor we consider below.

As the leading term of  $[I \setminus i_k | J \setminus s]^{(t-1)}$  divides  $\mathbf{y}^M$  but contains no critical coordinate, this quantum minor is not in  $K_{t-1}$ . Hence we may consider the supremum

$$\tilde{\mathcal{Q}} = (\tilde{Q}_1, \tilde{Q}_2, \dots, \tilde{Q}_{k-1})$$

and infimum

$$\mathcal{Q} = (Q_1, Q_2, \dots, Q_{k-1})$$

of  $\Gamma_B^{(t-1)}(I \setminus i_k | J \setminus s)$ .

Since  $(i_k, j'_k)$  is critical for some  $j'_k$ , there exists, by Claim 1, a critical quantum minor  $[I' | J']^{(t-1)}$  where, for an integer  $g$ , we write

$$I' = (i'_g < i'_{g+1} < \dots < i'_k = i_k) \quad \text{and} \quad J' = (j'_g < j'_{g+1} < \dots < j'_k).$$

Consider the supremum

$$\tilde{\mathcal{P}} = (\tilde{P}_g, \dots, \tilde{P}_k)$$

and infimum

$$\mathcal{P} = (P_g, \dots, P_k)$$

of  $\Gamma_B^{(t)}(I' | J')$ . As  $[I' | J']^{(t-1)}$  is critical,  $P_k$  is a path from  $i'_k$  to  $j'_k$  for which  $(r,s)$  is a J-turn.

Our ultimate goal is to show that we may always construct either a vertex-disjoint path system

$$\mathcal{R}_1 \in \Gamma_B^{(t-1)}(I | J),$$

or a vertex-disjoint path system

$$\mathcal{R}'_g \in \Gamma_B^{(t-1)}(I' | J').$$

In either case, we note that this contradicts the fact that the respective minor is in  $K_{t-1}$ , and so completes the proof of Claim 4. We proceed inductively.

We first construct  $\mathcal{R}_k$ . Recall that  $P_k$  has a subpath starting at  $i'_k = i_k$  and ending at  $(r, s)$ . Define  $Q_k$  to be this subpath followed by the vertical path from  $(r, s)$  to column vertex  $s$ . For purposes of induction, set  $v_k^0 = i_k$ ,  $v_k^1 = (r, s)$ , and note that  $v_k^0$  is the first vertex that  $P_k$  and  $Q_k$  have in common, while  $v_k^1$  is the last vertex they have in common. If one sets  $R_k = Q_k$ , then note that (see Figure 14) we trivially have:  $R_k = Q_k$  from  $i_k$  to  $v_k^0$ ;  $R_k = U(P_k, Q_k)$  from  $v_k^0$  to  $v_k^1$ ; and  $R_k = Q_k$  from  $v_k^1$  to  $j_k = s$ .

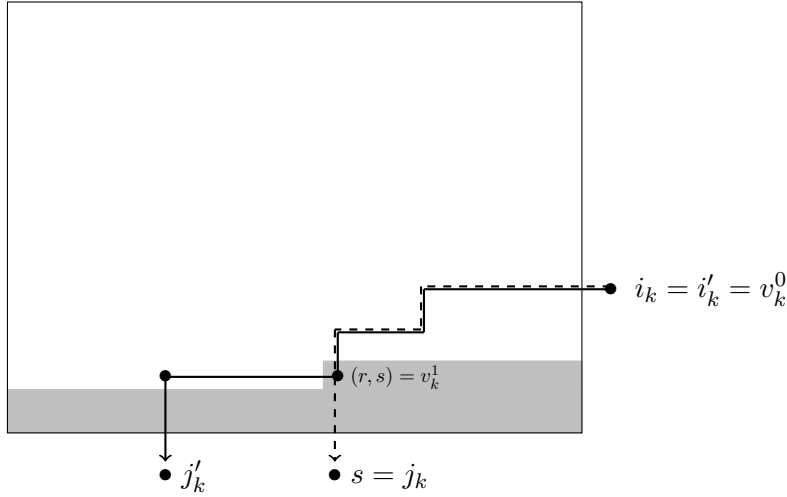


FIGURE 14. Construction of  $Q_k$  (dashed) from  $P_k$  (solid) in the proof of Claim 3.

Set  $\mathcal{R}_k = \{R_k\}$ . Of course,  $\mathcal{R}_k$  is a vertex-disjoint path system from  $i_k$  to  $j_k$ . If  $k = 1$ , then we have obtained the desired construction and hence contradiction. Therefore, suppose  $k > 1$ .

To construct  $\mathcal{R}'_k$ , we first show that  $j_{k-1} \geq j'_k$ . To the contrary, suppose  $j_{k-1} < j'_k$ , and consider

$$[I | J \setminus s \cup j'_k]^{(t-1)}.$$

If  $[I | J \setminus s \cup j'_k]^{(t-1)} \in K_{t-1}$ , then it is critical and so there exists a vertex-disjoint path system from  $I$  to  $J \setminus s \cup j'_k$  with the path from  $i_k$  to  $j'_k$  containing a J-turn at  $(r, s)$ . But as with the construction of  $Q_k$  above, we may replace this path with a path from  $i_k$  to  $s$ , thereby producing a vertex-disjoint path system from  $I$  to  $J$ , contradicting the choice of  $[I | J]^{(t-1)}$ .

The other possibility is that  $[I | J \setminus s \cup j'_k]^{(t-1)} \notin K_t$ , so that there does exist a vertex-disjoint path system from  $I$  to  $J \setminus s \cup j'_k$  where the path

$Q' : i_k \rightarrow j'_k$  does not contain a I-turn at  $(r, s)$ . We may assume this path system is

$$(\tilde{Q}_1, \dots, \tilde{Q}_{k-1}, Q').$$

Notice that  $\tilde{Q}_{k-1}$  is disjoint from  $Q'$ , and hence disjoint from  $L(Q', P_k)$  by a lemma similar to Lemma 4.1.13. But this latter path contains  $(r, s)$  (since  $P_k$  does) and so, as above, we may replace  $Q'$  with a path  $i_k$  to  $s$  thereby producing a vertex-disjoint path system from  $I$  to  $J$ , another contradiction. This completes the proof showing that  $j_{k-1} \geq j'_k$ .

Since  $k > 1$ , we may consider  $Q_{k-1}$ , which does not contain  $(r, s)$ . Notice that  $Q_{k-1}$  must intersect  $Q_k$  at a vertex coming before  $(r, s)$  on  $Q_k$ , as otherwise  $Q \cup Q_k$  is a vertex-disjoint path system from  $I$  to  $J$ . Let  $w_k^0$  be the first such common vertex. On the other hand, since  $j_{k-1} \geq j'_k$  and  $Q_{k-1}$  goes above  $(r, s)$ ,  $Q_{k-1}$  must share a vertex with  $P_k$  after  $(r, s)$ . Let  $w_k^1$  be the last vertex that  $Q_{k-1}$  and  $P_k$  share. See Figure 15.

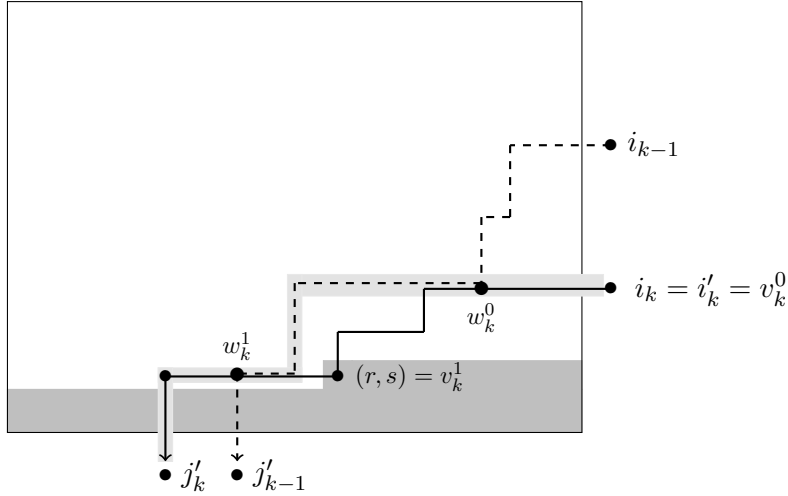


FIGURE 15.  $Q_{k-1}$  is the dashed path,  $P_k$  is the solid path,  $R'_k$  is the shadowed path.

Define  $R'_k$  to be the path that equals  $P_k$  from  $i'_k$  to  $w_k^0$ , then equals  $U(Q_{k-1}, P_k)$  from  $w_k^0$  to  $w_k^1$ , and then equals  $P_k$  from  $w_k^1$  to  $j'_k$ . See Figure 15. Observe that  $R'_k$  does not contain  $(r, s)$ , so that

$$\mathcal{R}'_k = \{R'_k\}$$

is a vertex-disjoint path system in  $\Gamma_B^{(t-1)}(i'_k | j'_k)$ . If  $k = g$ , then again we have obtained the desired contradiction, and so we may assume  $k > g$ .

Fix an integer  $\ell$  with  $\max(g, 1) \leq \ell < k$ . Suppose we know that  $i_{\ell+1} \leq i'_{\ell+1}$ ,  $j_\ell \geq j'_{\ell+1}$  and that we have constructed vertex-disjoint path systems  $\mathcal{R}_{\ell+1}$  and  $\mathcal{R}'_{\ell+1}$  as follows.

First,

$$\mathcal{R}_{\ell+1} = (R_{\ell+1}, \dots, R_k) \in \Gamma_B^{(t-1)}(i_{\ell+1}, \dots, i_k \mid j_{\ell+1}, \dots, j_k)$$

and there exists a vertex  $v_{\ell+1}^0$  which is the first vertex that  $P_{\ell+1}$  and  $Q_{\ell+1}$  have in common, a vertex  $v_{\ell+1}^1$  which is the last vertex that  $P_{\ell+1}$  and  $Q_{\ell+1}$  have in common, and that  $R_{\ell+1}$  equals  $Q_{\ell+1}$  from  $i_{\ell+1}$  to  $v_{\ell+1}^0$ , equals  $U(P_{\ell+1}, Q_{\ell+1})$  from  $v_{\ell+1}^0$  to  $v_{\ell+1}^1$ , and equals  $Q_{\ell+1}$  from  $v_{\ell+1}^1$  to  $j_{\ell+1}$ .

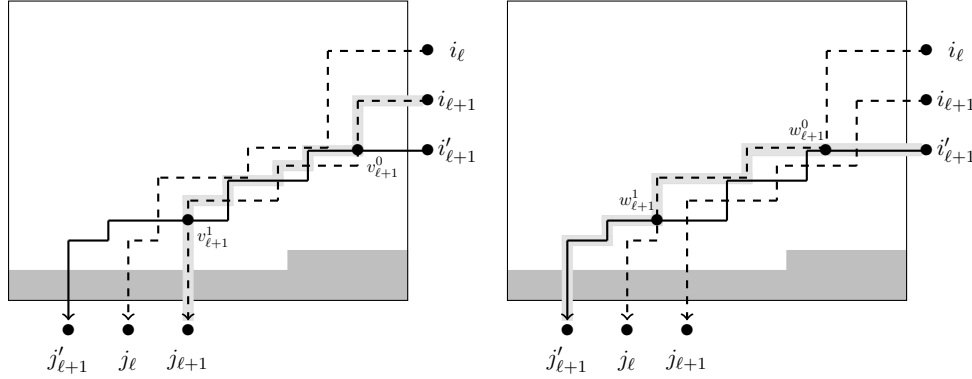


FIGURE 16.  $R_{\ell+1}$  is shaded path on the left diagram;  $R'_{\ell+1}$  is shaded path on the right diagram.

Second,

$$\mathcal{R}'_{\ell+1} = (R'_{\ell+1}, \dots, R'_k) \in \Gamma_B^{(t-1)}(i'_{\ell+1}, \dots, i'_k \mid j'_{\ell+1}, \dots, j'_k)$$

and there exists a vertex  $w_{\ell+1}^0$  which is the first vertex that  $P_{\ell+1}$  and  $Q_{\ell}$  have in common, a vertex  $w_{\ell+1}^1$  which is the last vertex that  $P_{\ell+1}$  and  $Q_{\ell}$  have in common, and that  $R'_{\ell+1}$  equals  $P_{\ell+1}$  from  $i'_{\ell+1}$  to  $w_{\ell+1}^0$ , equals  $U(P_{\ell+1}, Q_{\ell})$  from  $w_{\ell+1}^0$  to  $w_{\ell+1}^1$ , and equals  $P_{\ell+1}$  from  $w_{\ell+1}^1$  to  $j'_{\ell+1}$ .

We will construct a path  $R_{\ell} : i_{\ell} \rightarrow j_{\ell}$  disjoint from  $R_{\ell+1}$ , but first we need to show that  $i_{\ell} \leq i'_{\ell}$ . Suppose that  $i_{\ell} > i'_{\ell}$ . Since  $j_{\ell} \geq j'_{\ell+1} > j'_{\ell}$ , we may consider the minor

$$[i'_1, \dots, i'_{\ell}, i_{\ell}, \dots, i_{k-1} \mid j'_1, \dots, j'_{\ell}, j_{\ell}, \dots, j_{k-1}]^{(t-1)}.$$

Since this minor does not contain a critical coordinate and its leading term divides  $\mathbf{y}^M$ , we know that it is not in  $K_{t-1}$  and thus

$$\Gamma_B^{(t-1)}(i'_1, \dots, i'_{\ell}, i_{\ell}, \dots, i_{k-1} \mid j'_1, \dots, j'_{\ell}, j_{\ell}, \dots, j_{k-1}) \neq \emptyset.$$

Indeed,  $(\tilde{P}_1, \dots, \tilde{P}_{\ell}, Q_{\ell}, \dots, Q_{k-1})$  is a path system in this set, since for any path system we choose, the sub-path system from  $\{i'_1, \dots, i'_{\ell}\}$  to  $\{j'_1, \dots, j'_{\ell}\}$  may be replaced with the supremum of

$$\Gamma_B^{(t-1)}(i'_1, \dots, i'_{\ell} \mid j'_1, \dots, j'_{\ell}),$$

$$\Gamma_B^{(t-1)}(i_\ell, \dots, i_{k-1} \mid j_\ell, \dots, j_{k-1}).$$
$$\{\bar{P}_1, \dots, \bar{P}_\ell\} \cup \mathcal{R}'_{\ell+1}$$

Next, observe that  $P_\ell$  must also intersect  $Q_\ell$  at a vertex coming after  $w_{\ell+1}^0$ , as otherwise,  $P_\ell$  is disjoint from  $R'_{\ell+1}$  after  $w_{\ell+1}^0$ . By the construction of  $R'_{\ell+1}$ , we then must have  $(P_1, \dots, P_\ell) \cup \mathcal{R}'_\ell$  is a vertex-disjoint path system

in  $\Gamma_B^{(t-1)}(I' | J')$ . Let  $v_\ell^1$  be the last vertex that  $Q_\ell$  and  $P_\ell$  have in common. Define  $R_\ell$  as the path equal to  $Q_\ell$  from  $i_\ell$  to  $v_\ell^0$ , equal to  $U(P_\ell, Q_\ell)$  from  $v_\ell^0$  to  $v_\ell^1$ , and then equal to  $Q_\ell$  from  $v_\ell^1$  to  $j_\ell$ . Since  $Q_\ell$  is disjoint from  $Q_{\ell+1}$  up to  $v_\ell^0$  and after  $v_\ell^1$ , and  $U(P_\ell, Q_\ell)$  is disjoint from  $U(P_{\ell+1}, Q_{\ell+1})$ , we see that  $R_\ell$  is disjoint from  $R_{\ell+1}$ , and so

$$\mathcal{R}_\ell = \mathcal{R}_{\ell+1} \cup R_\ell \in \Gamma_B^{(t-1)}(i_\ell, \dots, i_k | j_\ell, \dots, j_k).$$

If  $\ell = 1$ , then we have obtained the required path system completing the proof of this claim.

Assume  $\ell > 1$ . To construct  $\mathcal{R}'_\ell$ , we first must show that  $j_{\ell-1} \geq j'_\ell$ . Suppose, to the contrary, that  $j_{\ell-1} < j'_\ell$ . Now,  $i_{\ell-1} < i_\ell \leq i'_\ell$ , so we may consider the minor

$$[i_1, \dots, i_{\ell-1}, i'_\ell, \dots, i'_k | j_1, \dots, j_{\ell-1}, j'_\ell, \dots, j'_k]^{(t-1)}.$$

Since the leading term of this minor divides  $\mathbf{y}^M$ , there are two possibilities. If it is an element of  $K_{t-1}$ , then it is a critical minor, and so there is a vertex-disjoint path system in

$$\Gamma_B^{(t)}(i_1, \dots, i_{\ell-1}, i'_\ell, \dots, i'_k | j_1, \dots, j_{\ell-1}, j'_\ell, \dots, j'_k),$$

which we may take to be

$$(\tilde{Q}_1, \dots, \tilde{Q}_{\ell-1}, P_\ell, \dots, P_k).$$

Therefore,  $\tilde{Q}_{\ell-1}$  is disjoint from both  $P_\ell$  and  $Q_\ell$ , and so disjoint from  $R_\ell$  by the latter path's construction. Hence,  $(\tilde{Q}_1, \dots, \tilde{Q}_{\ell-1}) \cup \mathcal{R}_\ell$  is a vertex-disjoint path system in the empty set  $\Gamma_B^{(t-1)}(I | J)$ , an impossibility. The other possibility is that the quantum minor is not in  $K_{t-1}$ . This possibility is dealt with similarly to the corresponding case in the proof above that  $j_{k-1} \geq j'_k$ . It follows that  $j_{\ell-1} \geq j'_\ell$ .

We now describe the construction of  $R'_\ell$ . Since  $\ell > 1$ , consider  $Q_{\ell-1}$ . This path is disjoint from  $Q_\ell$ . If  $Q_{\ell-1}$  does not intersect  $P_\ell$  at a vertex between  $v_\ell^0$  and  $v_\ell^1$ , then  $Q_{\ell-1}$  is disjoint from  $R_\ell$  so that  $(Q_1, \dots, Q_{\ell-1}) \cup \mathcal{R}_\ell$  is a vertex-disjoint path system in the empty set  $\Gamma_B^{(t-1)}(I | J)$ , an impossibility. So we may let  $w_\ell^0$  be the first vertex that  $Q_{\ell-1}$  shares with  $P_\ell$ . Now, since  $j'_\ell \leq j_{\ell-1} < j_\ell$ , and the two subpaths of  $P_\ell$  and  $Q_\ell$  starting at  $v_\ell^1$ , together with the line from  $j'_\ell$  to  $j_\ell$  is a closed curve in the plane,  $Q_{\ell-1}$  must intersect  $P_\ell$  at a vertex after  $v_\ell^1$ . Let  $w_\ell^1$  be their last common vertex after  $v_\ell^1$ . We now take  $R'_\ell$  to be the path equal to  $P_\ell$  from  $i'_\ell$  to  $w_\ell^0$ ; equal to  $U(P_\ell, Q_{\ell-1})$  from  $w_\ell^0$  to  $w_\ell^1$ ; and equal to  $P_\ell$  from  $w_\ell^1$  to  $j'_\ell$ . See Figure 18 for an example. That  $R'_\ell$  is disjoint from  $R'_{\ell+1}$  is seen similarly as when we showed that  $R_\ell$  and  $R_{\ell+1}$  are disjoint.

Of course, we now take

$$\mathcal{R}'_\ell = \mathcal{R}'_{\ell+1} \cup R'_\ell \in \Gamma_B^{(t-1)}(i'_\ell, \dots, i'_k | j'_\ell, \dots, j'_k).$$

If  $\ell = g$ , then we are done. Otherwise continue as above. As this process cannot continue indefinitely, we see that eventually we can construct either

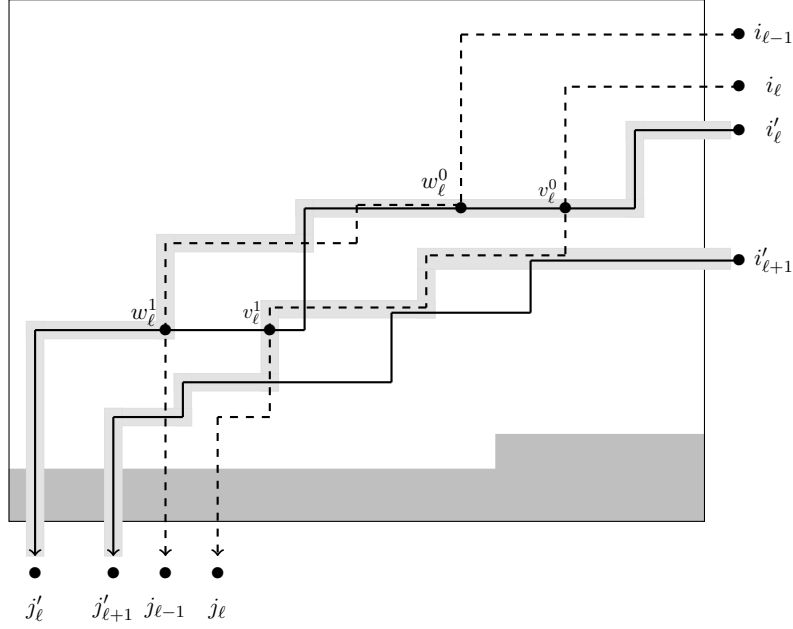


FIGURE 18. Constructing  $R'_\ell$  (upper shaded path). Note that it is disjoint from  $R'_{\ell+1}$  (lower shaded path).

a vertex-disjoint path system from  $I$  to  $J$  or a vertex-disjoint path system from  $I'$  to  $J'$ , either construction providing the required contradiction. This completes the proof of Claim 3.

**Claim 4.** *The term  $\mathbf{y}^{N_C}$  appears in the lexicographic expression of  $b = \overleftarrow{a} y_{r,s}^h$  with a nonzero coefficient.*

*Proof of Claim 4:* Recall that a lexicographic term is said to *appear* in an element of  $A^{(t-1)}$  or  $A^{(t)}$  if it has a nonzero coefficient in the lexicographic expression of that element.

We have already seen that  $\mathbf{y}^{N_C}$  appears in

$$\overleftarrow{x}_{i_1, j_1}^C \overleftarrow{x}_{i_2, j_2}^C \cdots \overleftarrow{x}_{i_p, j_p}^C y_{r,s}^h.$$

We will show that this is, in fact, the unique appearance of  $\mathbf{y}^{N_C}$  in (the lexicographic expression of) any summand of

$$b = \overleftarrow{a} y_{r,s}^h = \overleftarrow{x}^M y_{r,s}^h + \sum_L \alpha_L \overleftarrow{x}^L y_{r,s}^h,$$

and so appears in  $b$ .



To start, consider in  $\overleftarrow{\mathbf{x}}^M y_{r,s}^h$  the lexicographic expression of some

$$\overleftarrow{x}_{i_1,j_1}^{C'} \overleftarrow{x}_{i_2,j_2}^{C'} \cdots \overleftarrow{x}_{i_p,j_p}^{C'} y_{r,s}^h = \sum_{L_{C'} \in \mathcal{M}_{m,n}(\mathbb{Z})} \alpha_{L_{C'}} \mathbf{y}^{L_{C'}},$$

where  $C' \neq C$ . Suppose  $C'$  is chosen so that there is an  $L_{C'}$  equal to  $N_C$ .

Now, by Lemma 4.3.1, each term  $\mathbf{y}^{L_{C'}}$  satisfies  $(L_{C'})_{r,s} = h - |C'|$ . Since  $(N_C)_{r,s} = h - |C|$ , we must have if  $|C'| = |C| > 0$ . But, since  $C \neq C'$ , there must exist  $k \in C'$  such that  $(i_k, j_k)$  is not a critical coordinate. Since  $(i_k, j_k)$  is not critical, we should have

$$\begin{aligned} (L_{C'})_{i_k,j_k} &= (N_C)_{i_k,j_k} \\ &= (M)_{i_k,j_k} \\ &> (M)_{i_k,j_k} - |\{k' \in C' \mid (i_{k'}, j_{k'}) = (i_k, j_k)\}|. \end{aligned}$$

By Part 4 of Lemma 4.3.1, there is a coordinate  $(i_k, j)$  with  $j < j_k$  and

$$\begin{aligned} (L_{C'})_{i_k,j} &< (M)_{i_k,j} \\ &= (N_C)_{i_k,j}, \end{aligned}$$

where the equality follows from the fact that since  $(i_k, j_k)$  is not critical, neither is  $(i_k, j)$  by Claim 2. Hence,  $L_{C'}$  cannot be equal to  $N_C$  since their entries differ in coordinate  $(i_k, j)$ . This is a contradiction and so we conclude that  $\mathbf{y}^{N_C}$  appears in  $\overleftarrow{\mathbf{x}}^M y_{r,s}^h$ .

Next, suppose

$$\mathbf{x}^L = x_{a_1,b_1} \cdots x_{a_t,b_t},$$

appears in  $a$ , where  $(a_k, b_k) \leq (a_{k+1}, b_{k+1})$  for each  $k \in [t-1]$ , and where  $L \prec M$  at coordinate  $(i, j)$ . With the notation of Section 4.3, consider

$$\overleftarrow{\mathbf{x}}^L y_{r,s}^h = \sum_D q^{|D|} \overleftarrow{x}_{a_1,b_1}^D \overleftarrow{x}_{a_2,b_2}^D \cdots \overleftarrow{x}_{a_t,b_t}^D y_{r,s}^h.$$

Suppose that  $\mathbf{y}^{N_C}$  appears in

$$\overleftarrow{x}_{a_1,b_1}^D \overleftarrow{x}_{a_2,b_2}^D \cdots \overleftarrow{x}_{a_t,b_t}^D y_{r,s}^h = \sum_{L_D} \alpha_{L_D} \mathbf{y}^{L_D}.$$

By Lemma 4.3.1, Part 5, every entry in an  $L_D$  with coordinates not northwest, north or west of  $(r, s)$  must equal the corresponding entry in  $L$ . Since we also require  $L_D = N_C$  for some  $D$ , this implies that those entries are equal to the corresponding entry in  $M$  as well. Thus,  $(i, j)$  can only be north, west or northwest of  $(r, s)$ . On the other hand, if  $j = s$ , then all entries in  $L$  and  $M$  in row  $i$  except coordinate  $(i, j)$  are equal. By homogeneity, this means that we must also have  $(L)_{i,j} = (M)_{i,j}$ , a contradiction. Hence  $(i, j)$  is not north of  $(r, s)$ , and by similar reasoning  $(i, j)$  is not west of  $(r, s)$ . Therefore, we may assume that  $L \prec M$  at a coordinate  $(i, j)$  northwest of  $(r, s)$ .

There are two cases to consider. First, suppose  $(i, j)$  is not a critical coordinate. In this case,

$$(N_C)_{i,j} = (M)_{i,j} > (L)_{i,j},$$

and so we may proceed as above by applying Part 4 of Lemma 4.3.1 to see that in order to have  $(L_D)_{i,j} = (N_C)_{i,j}$ , we would require an entry with coordinate  $(i, j')$  with  $j' < j$  to satisfy

$$\begin{aligned} (L_D)_{i,j'} &< (L)_{i,j'} \\ &= (M)_{i,j'} \\ &= (N_C)_{i,j'}. \end{aligned}$$

Hence we cannot have  $N_C = L_D$  in this case.

Next, suppose  $(i, j)$  is critical. Let  $(i, j_0)$  be the least critical coordinate in row  $i$ . Notice that no  $(i, j') = (a_k, b_k)$  with  $j' < j_0$  has  $k \in D$ , for reasons similar to the previous paragraph. Now, consider  $j'$  where  $j_0 < j' \leq s$ . By Part 3 of Claim 1 applied to  $(i, j_0)$ , we know that every entry of  $M$  south of  $(i, j')$  is equal to zero. Hence, the sum of the entries in column  $j'$  of  $M$  is equal to  $\sum_{i'=1}^i (M)_{i',j'}$ . By homogeneity, this is equal to the sum of the entries in column  $j'$  of  $L$ . On the other hand, the entries north of  $(i, j')$  in  $L$  are equal to the corresponding entries in  $M$ . Since all entries of  $L$  are nonnegative, we see that

$$(L)_{i,j'} \leq (M)_{i,j'},$$

for every  $j_0 < j' \leq s$ . Also, since the entries of  $L$  and  $M$  are equal prior to  $(i, j_0)$  and  $L \prec M$ , we must also have  $(L)_{i,j_0} \leq (M)_{i,j_0}$ . But, since we know that  $(L)_{i,j} < (M)_{i,j}$ , applying Part 3 of Lemma 4.3.1 gives

$$\begin{aligned} (L_D)_{i,s} &= (L)_{i,s} + |\{k \in D \mid i_k = i\}| \\ &\leq (L)_{i,s} + \sum_{j'=j_0}^s (L)_{i,j'} \\ &< (M)_{i,s} + \sum_{j'=j_0}^s (M)_{i,j'} \\ &= (N_C)_{i,s}. \end{aligned}$$

Hence, we cannot have  $L_D = N_C$  in this case either, and so this completes the proof of Claim 4.

**Claim 5.** *There exists an element of  $K_{t-1}$  for which  $\mathbf{y}^{N_C}$  is the leading term.*

Note that Claims 3 and 5 are incompatible, thus providing the required contradiction to the assumptions on the entries of  $M$  and completing the proof of Theorem 4.4.1.

*Proof of Claim 5:* By Lemma 3.3.4, we may write

$$b = \sum_{i=0}^{\infty} b_i y_{r,s}^i,$$

where finitely many  $b_i \neq 0$  and each  $b_i \in K_{t-1}$  with lexicographic expression using only generators with coordinates less than  $(r, s)$ .

By Claim 4,  $\mathbf{y}^{N_C}$  appears in  $b$  and so, since  $(N_C)_{r,s} = h - |C|$ , it appears in

$$z_0 = b_{h-|C|} y_{r,s}^{h-|C|}.$$

Suppose, for a positive integer  $k$ , that we have constructed an element  $z_{k-1} \in K_{t-1}$  in which  $\mathbf{y}^{N_C}$  appears. Moreover, suppose any term appearing in  $z_{k-1}$  that is greater than  $\mathbf{y}^{N_C}$ , also appears in  $z_0$ . If  $\ell t(z_{k-1}) = \mathbf{y}^{N_C}$ , then we have found the required element of  $K_{t-1}$ . Otherwise, we construct below an element  $z_k \in K_{t-1}$  with the same above properties as  $z_{k-1}$ , but in which strictly less terms greater than  $\mathbf{y}^{N_C}$  appear. Since there are only finitely many terms appearing in  $z_0$  that are greater than  $\mathbf{y}^{N_C}$ , this process must end after finitely many steps, resulting in an element of  $K_{t-1}$  whose leading term is  $\mathbf{y}^{N_C}$ , as required.

Let

$$\ell t(z_{k-1}) = \mathbf{y}^L \succ \mathbf{y}^{N_C},$$

so that for some  $\gamma_L, \gamma_{N_C} \in \mathbb{K}^*$  we may write

$$z_{k-1} = \gamma_L \mathbf{y}^L + \gamma_{N_C} \mathbf{y}^{N_C} + z'_{k-1}.$$

In particular, observe that in  $z'_{k-1}$ , there appear strictly less terms greater than  $\mathbf{y}^{N_C}$  than in  $z_{k-1}$ . Also,  $\mathbf{y}^L \prec \mathbf{y}^M y_{r,s}^h$  since the latter term is the leading term of  $b$  but  $\mathbf{y}^L \in b_{h-|C|} y_{r,s}^{h-|C|} \neq b_h y_{r,s}^h$  since  $|C| > 0$ . Finally, for  $i \in [r-1]$ , let  $C_i$  denote the critical coordinates in row  $i$ .

Let  $i_0$  be the least index such that  $C_i = C_{i_0}$  is non-empty. Let  $(c_0, d_0)$  be the least coordinate in  $C_{i_0}$ . Since  $\mathbf{y}^{N_C} \prec \mathbf{y}^L \prec \mathbf{y}^M y_{r,s}^h$  and the entries of  $N_C$  and  $M$  at coordinates prior to  $(c_0, d_0)$  are equal, we have that the entries of  $L$ ,  $M$  and  $N_C$  are equal prior to  $(c_0, d_0)$  as well.

Suppose  $(c_0, d) \in C_{i_0}$  is such that  $(L)_{c_0, d} > 0$ . In this case, we proceed as follows. Since  $(c_0, d)$  is a critical coordinate, there is a critical minor  $[I | J]^{(t-1)} \in K_{t-1}$  with maximum coordinate  $(c_0, d)$  whose leading term divides  $\mathbf{y}^M$ , and so divides  $\mathbf{y}^L$  by the previous paragraph. By Lemma 4.2.8, we have

$$\mathbf{y}^L = q^\alpha [I | J]^{(t-1)} \mathbf{y}^{L-P_{\text{Id}}} + w,$$

where  $w \in A^{(t-1)}$  has the property that if  $\ell t(w) = \mathbf{y}^K$ , then  $K \prec L$  at an entry northwest of  $(c_0, d)$ . Since all entries of  $L$  northwest of  $(c_0, d)$  are equal to those of  $N_C$  and  $M$ , we have that  $\ell t(w) \prec \mathbf{y}^{N_C}$  as well.

Hence,

$$\begin{aligned} z_{k-1} &= \gamma_L \mathbf{y}^L + \gamma_{N_C} \mathbf{y}^{N_C} + z'_{k-1} \\ &= \gamma_L (q^\alpha [I | J]^{(t-1)} \mathbf{y}^{L-P_{\text{id}}} + w) + \gamma_{N_C} \mathbf{y}^{N_C} + z'_{k-1}, \end{aligned}$$

so that if we define

$$\begin{aligned} z_k &= z_{k-1} - \gamma_L q^\alpha [I | J]^{(t-1)} \mathbf{y}^{L-P_{\text{id}}} \\ &= \gamma_{N_C} \mathbf{y}^{N_C} + \gamma_L w + z'_{k-1}, \end{aligned}$$

then we have  $z_k \in K_{t-1}$  satisfying the desired properties described above.

Now, suppose each coordinate  $(c_0, d) \in C_{i_0}$  is such that  $(L)_{c_0, d} = 0$ . Thus,  $L$  and  $N_C$  are equal in all entries prior to  $(c_0, s)$ . Also, since  $\mathbf{y}^L$  appears in  $b$ , there must be a  $\mathbf{x}^{L'}$  appearing in  $a$  so that  $\mathbf{y}^L$  appears in  $\overleftarrow{\mathbf{x}^{L'}} y_{r,s}^h$ . We also have  $\mathbf{x}^{L'} \preceq \mathbf{x}^M$ , and it follows by Part 2 of Lemma 4.3.1, that the entries in  $L'$  and  $M$  are equal prior to  $(c_0, d_0)$ .

Now, as in the proof of Claim 4, we may apply homogeneity to conclude that  $(L')_{c_0, d} \leq (M)_{c_0, d}$  for each  $(c_0, d) \in C_{i_0}$ , and if any of these inequalities are strict, then  $(L)_{i_0, s} < (N_C)_{i_0, s}$ , contradicting the assumption that  $N_C \prec L$ . Hence,  $L'$  and  $M$  have equal entries prior to  $(c_0, s)$ .

Now, let  $i_1$  be the second least index such that  $C_{i_1}$  is nonempty, and consider coordinates from  $(c_0, s)$  to  $(c_1, d_1)^-$ , where  $(c_1, d_1)$  is the least coordinate in  $C_{i_1}$ . Since  $\mathbf{y}^{N_C} \prec \mathbf{y}^L$ , we know that if any entry in  $L$  and  $N_C$  in these coordinates differ, then the first differing entry is larger in  $L$  than in  $N_C$ . On the other hand, the entries of  $N_C$  and  $M$  are equal in this range of coordinates. Thus, if the first differing entry is larger in  $L$  than in  $N_C$ , then this entry in  $L'$  is larger than in  $M$ , yet every entry prior in  $L'$  is equal to that in  $M$ , implying that  $\mathbf{y}^M \prec \mathbf{y}^{L'}$ , a contradiction. Hence, the entries in this range of coordinates are equal in  $N_C, M, L$  and  $L'$ .

Since all entries north-west of a critical coordinate are equal in  $M, N_C, L$  and  $L'$ , we may now repeat the above arguments with the coordinates in  $C_{i_1}$ , and subsequent  $C_i$ 's if necessary. Eventually we must find a critical coordinate with a positive entry in  $L$ , as otherwise we would find that  $N_C = L$ , contradicting the assumption that  $\mathbf{y}^{N_C} \prec \mathbf{y}^L$ . Hence, we can always construct the required  $z_k$  and, eventually, an element of  $K_{t-1}$  with leading term  $\mathbf{y}^{N_C}$ . This completes the proof of Claim 5 and the theorem.  $\square$

**4.5. Conclusions.** As an immediate consequence of Theorem 4.4.1, we see that if  $q$  is a non-root of unity, then every  $\mathcal{H}$ -prime of  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$  is generated, as a left and right ideal, by the quantum minors it contains. We can sharpen this result by finding a *minimal* Gröbner basis. The idea here is simple: if  $G$  is a Gröbner basis for some ideal and if  $g_1, g_2 \in G$  are such that  $\ell t(g_2)$  divides  $\ell t(g_1)$ , then  $G \setminus g_1$  remains a Gröbner basis for the ideal. In other words, we may throw out any redundant elements in  $G$ . With respect to  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ , this means the following. Suppose  $[I | J]^{(mn)}$  is a minor

where  $I = \{i_1 < i_2 < \cdots < i_k\}$  and  $J = \{j_1 < j_2 < \cdots < j_k\}$ . If  $L \subsetneq [k]$ ,  $I' = I \cap \{i_\ell \mid \ell \in L\}$  and  $J' = J \cap \{j_\ell \mid \ell \in L\}$ , then we call  $[I' \mid J']^{(mn)}$  a *diagonal subminor* of  $[I \mid J]^{(mn)}$ . Theorem 4.4.1 immediately implies the following, a strengthened version of the Goodearl-Lenagan conjecture.

**Corollary 4.5.1.** *Let  $q$  be a nonzero, non-root of unity and let  $K$  be an  $\mathcal{H}$ -prime in  $\mathcal{O}_q(\mathcal{M}_{m,n}(\mathbb{K}))$ . Then  $K$  is generated, as a left and right ideal, by those quantum minors in  $K$  with no diagonal subminor in  $K$ .*

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#### REFERENCES

- [1] K. A. Brown and K. R. Goodearl, *Lectures on algebraic quantum groups*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2002.
- [2] J. L. Bueso, J. Gómez-Torrecillas, and A. Verschoren, *Algorithmic methods in non-commutative algebra: Applications to quantum groups*, Kluwer University Press, 2003.
- [3] K. Casteels, *A graph theoretic method for determining generating sets of prime ideals in quantum matrices*, Journal of Algebra **330** (2011), 188–205.
- [4] G. Cauchon, *Effacement des dérivations et spectres premiers des algèbres quantiques*, J. Algebra **260** (2003), no. 2, 476–518.
- [5] ———, *Spectre premier de  $\mathcal{O}_q(M_n(k))$ : image canonique et séparation normale*, J. Algebra **260** (2003), no. 2, 519–569.
- [6] J. Geiger and M. Yakimov, *Quantum schubert cells via representation and ring theory*, <http://arxiv.org/abs/1203.3780>, 2012.
- [7] K. R. Goodearl, S. Launois, and T. H. Lenagan, *Torus-invariant prime ideals in quantum matrices, totally nonnegative cells and symplectic leaves*, Math. Z. **269** (2011), no. 1-2, 29–45.
- [8] ———, *Totally nonnegative cells and matrix poisson varieties*, Advances in Mathematics **226** (2011), no. 1, 779–826.
- [9] K. R. Goodearl and T. H. Lenagan, *Prime ideals invariant under winding automorphisms in quantum matrices*, Internat. J. Math. **13** (2002), no. 5, 497–532.
- [10] ———, *Winding-invariant prime ideals in quantum  $3 \times 3$  matrices*, J. Algebra **260** (2003), no. 2, 657–687.
- [11] K. R. Goodearl and E. S. Letzter, *Prime and primitive spectra of multiparameter quantum affine spaces*, Trends in ring theory (Miskolc, 1996), CMS Conf. Proc., vol. 22, Amer. Math. Soc., Providence, RI, 1998, pp. 39–58.
- [12] ———, *The Dixmier-Moeglin equivalence in quantum coordinate rings and quantized Weyl algebras*, Trans. Amer. Math. Soc. **352** (2000), no. 3, 1381–1403.
- [13] T. Lam and L. Williams, *Total positivity for cominiscule grassmannians*, New York Journal of Mathematics **14** (2008), 53–99.
- [14] S. Launois, *Generators for  $\mathcal{H}$ -invariant prime ideals in  $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$* , Proc. Edinb. Math. Soc. (2) **47** (2004), no. 1, 163–190.
- [15] ———, *Les idéaux premiers invariants de  $\mathcal{O}_q(\mathcal{M}_{m,p}(\mathbb{C}))$* , J. Algebra **272** (2004), no. 1, 191–246.
- [16] B. Lindström, *On the vector representations of induced matroids*, Bull. London Math. Soc. **5** (1973), 85–90.
- [17] A. Postnikov, *Total positivity, grassmannians, and networks*, 2006, <http://arxiv.org/abs/0609764>.

- [18] Kelli Talaska, *Combinatorial formulas for le-coordinates in a totally nonnegative grassmannian*, Journal of Combinatorial Theory, Series A **118** (2011), no. 1, 58–66.
- [19] M. Yakimov, *A proof of the goedearl lenagan polynormality conjecture*, To Appear in Int. Math. Res. Not.
- [20] ———, *Invariant prime ideals in quantizations of nilpotent Lie algebras*, Proc. Lond. Math. Soc. (3) **101** (2010), no. 2, 454–476.

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